HARMONIC MAPS INTO THE EXCEPTIONAL SYMMETRIC SPACE $G_2/SO(4)$

MARTIN SVENSSON AND JOHN C. WOOD

ABSTRACT. We show that a harmonic map from a Riemann surface into the exceptional symmetric space $G_2/\mathrm{SO}(4)$ has a J_2-holomorphic twistor lift into one of the three flag manifolds of G_2 if and only if it is 'nilconformal', i.e., has nilpotent derivative. The class of nilconformal maps includes those of finite uniton number studied by N. Correia and R. Pacheco, however we exhibit examples which are not of finite uniton number. Then we find relationships with almost complex maps from a surface into the 6-sphere; this enables us to construct examples of nilconformal harmonic maps into $G_2/SO(4)$ which are not of finite uniton number, and which have lifts into any of the three twistor spaces.

1. INTRODUCTION

Harmonic maps are smooth maps between Riemannian manifolds which extremize the Dirichlet energy integral (see, for example, [12]). Harmonic maps from surfaces into symmetric spaces are of particular interest both to geometers, as they include *minimal surfaces*, and to theoretical physicists, as they constitute the *non-linear σ-model of particle physics*. Twistor methods for finding such harmonic maps have been around for some time: a general theory was given by F. E. Burstall and J. H. Rawnsley [6], see also [11]. The idea is to find a *twistor fibration (for harmonic maps)*: this is a fibration $Z \to N$ from an almost complex manifold Z, called a *twistor space*, to a Riemannian manifold N, with the property that (almost-)holomorphic maps from (Riemann) surfaces to Z project to harmonic maps into N. For a symmetric space $N = G/H$, the theory of [6] provides twistor spaces which are (generalized) flag manifolds of G precisely when N is *inner*, i.e., G and H have the same rank; those twistor spaces come equipped with canonical non-integrable complex structures J_2 and canonical fibration to N.

The exceptional symmetric space $G_2/\mathrm{SO}(4)$ has precisely three twistor spaces T_s $(s = 1, 2, 3)$ which are flag manifolds of G_2, see §2.3.

Throughout this paper, by *surface* we shall mean a *Riemann surface*, i.e., a connected (not necessarily compact) one-dimensional complex manifold. A harmonic map from a surface is called *nilconformal* if its derivative is nilpotent, see Definition 3.2 below. Our main result is as follows:

Theorem 1.1. *A harmonic map from a surface to $G_2/\mathrm{SO}(4)$ has a twistor lift into one of the three twistor spaces T_s if and only if it is nilconformal.*

This was previously known only for harmonic maps from the 2-sphere [6]; these are all of finite unition number [27]. The class of nilconformal harmonic maps includes those of finite uniton number, but is strictly bigger, see Example 5.11. In [10], N. Correia and R. Pacheco studied harmonic maps of finite uniton number into G_2 and $G_2/\mathrm{SO}(4)$; the two approaches agree in the case of maps with superhorizontal twistor lift, see §4.5.

The method is as follows. In [26], the authors showed that a harmonic map from a surface to a compact classical inner symmetric space has a twistor lift if and only if it is nilconformal —

2000 *Mathematics Subject Classification.* 53C43, 58E20.

Key words and phrases. harmonic map, twistor, exceptional Lie group, non-linear sigma model.

The first author was supported by the Danish Council for Independent Research under the project *Symmetry Techniques in Differential Geometry*. The second author thanks the Department of Mathematics and Computer Science of the University of Southern Denmark, Odense, for support and hospitality during part of the preparation of this work.

this was previously only known for the complex Grassmannian [5]. As in [27], the derivative of a harmonic map $\varphi : M \to G$ from a surface to a Lie group defines an endomorphism A^φ_z of a trivial complex bundle over M; nilconformality means that this endomorphism is nilpotent. The lifts are constructed from this endomorphism, though except in simple cases, there is some degree of choice.

The exceptional symmetric space $G_2/\mathrm{SO}(4)$ is the space of all associative 3-dimensional subspaces in $\mathbb{R}^7$, so that we can embed $G_2/\mathrm{SO}(4)$ in the real Grassmannian $G_3(\mathbb{R}^7)$ and thence in $\mathrm{U}(7)$. Then we have the endomorphism A^φ_z of the trivial $\mathbb{C}^7$-bundle over M, which we use to construct our lifts. This time, our lifts are canonical and we have totally explicit formulae for them. By realizing the twistor spaces as flags satisfying a G_2-condition, we interpret our work in terms of the constructions in [26] in §4.4.

The authors thank Fran Burstall for several illuminating conversations.

2. Preliminaries

2.1. **The Octonions and the action of G_2.** Let $\mathbb{H} \cong \mathbb{R}^4$ denote the quaternions and $\mathrm{Im}\,\mathbb{H} \cong \mathbb{R}^3$ the space of imaginary quaternions, both with their canonical orientations. The algebra of *octonions* $\mathbb{O}$, or *Cayley numbers*, [17, p. 113*ff*] is an alternative real division algebra of dimension 8 given by $\mathbb{O} = \mathbb{H} \oplus \mathbb{H} \cdot e$ for some unit octonion e. Note that $\mathbb{O}$ and its imaginary part $\mathrm{Im}\,\mathbb{O} = \mathrm{Im}\,\mathbb{H} \oplus \mathbb{H} \cdot e$ acquire a canonical orientation from those of $\mathbb{H}$ and $\mathrm{Im}\,\mathbb{H}$.

The octonion multiplication $\cdot$ on $\mathbb{O} \cong \mathbb{R}^8$ induces a vector product $\times$ on $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$ by $v \times w =$ the imaginary part of $v \cdot w$. On the other hand, the real part of $v \cdot w$ gives a positive definite inner product $(\cdot,\cdot)$ on $\mathrm{Im}\,\mathbb{O}$; we extend both of these by complex bilinearity to $\mathrm{Im}\,\mathbb{O} \otimes \mathbb{C} \cong \mathbb{C}^7$. The *Hermitian inner product of* $v, w \in \mathrm{Im}\,\mathbb{O} \otimes \mathbb{C}$ is given by $(v, \overline{w})$; here, if $w = a + b\mathrm{i}$ $\;a, b \in \mathrm{Im}\,\mathbb{O}$, $\overline{w}$ denotes its complex conjugate $a - b\mathrm{i}$. If $(v, \overline{w}) = 0$, we call v and w *orthogonal*, written $v \perp w$.

The vector product is clearly bilinear and enjoys the following properties for $u, v, w \in \mathrm{Im}\,\mathbb{O} \otimes \mathbb{C}$ [17]:

$$(u, v \times w) = (u \times v, w), \tag{2.1}$$

$$u \times (v \times w) + (u \times v) \times w = 2(u, w)v - (u, v)w - (v, w)u\,. \tag{2.2}$$

A 3-dimensional subspace $\xi \subset \mathrm{Im}\,\mathbb{O}$ is said to be *associative* if it is the imaginary part of a subalgebra isomorphic to the quaternions. It acquires a canonical orientation from that of the quaternions. In the sequel, we write $\overline{L}_j$ to mean $\overline{L_j}$.

Lemma 2.1. *Let ξ be a 3-dimensional subspace of* $\mathrm{Im}\,\mathbb{O}$.

(a) *The following are equivalent*:

(i) *ξ is associative*;
(ii) *ξ is closed under the vector product*;
(iii) *ξ has an orthonormal basis $\{e_1, e_2, e_3\}$ with $e_1 \times e_2 = e_3$*;
(iv) *$\xi \otimes \mathbb{C}$ has a basis $\{\overline{L}_1, L_0, L_1\}$ with $L_0 = \overline{L}_0$ and $L_1 \times \overline{L}_1 = \mathrm{i}L_0$*;
(v) *$\xi \otimes \mathbb{C}$ has a basis $\{\overline{L}_1, L_0, L_1\}$ with $L_0 = \overline{L}_0$ and $L_0 \times L_1 = \mathrm{i}L_1$*.

(b) *The canonical orientation of an associative subspace ξ is that given by $\{e_1, e_2, e_1 \times e_2\}$ for any linearly independent elements e_1, e_2 of ξ. Then* (iii) *holds for* any *oriented orthonormal basis $\{e_i\}$, and* (iv) *and* (v) *hold for $L_0 = e_1$, $L_1 = (1/\sqrt{2})(e_2 - \mathrm{i}e_3)$.*

Proof. (a) Clearly (i) implies (ii) and (iii). Conversely, given (iii), applying (2.2), we obtain $e_2 \times e_3 = e_1$ and $e_3 \times e_1 = e_2$ so that $e_1 \mapsto \mathrm{i}$, $e_2 \mapsto \mathrm{j}$ defines an isomorphism with the imaginary quaternions, i.e., (i) holds; this also establishes part (b).

That (iii), (iv) and (v) are equivalent is seen by putting $L_0 = e_1$, $L_1 = (1/\sqrt{2})(e_2 - \mathrm{i}e_3)$ and using (2.2).

(b) This is clear. □

The group G_2 is the automorphism group of $\mathbb{O}$; it stabilizes 1, and since the scalar product is the real part of $\cdot$, it acts by isometries on $\mathrm{Im}\,\mathbb{O}$. Since it also preserves the vector product, it preserves

orientation so that $G_2 \subset \mathrm{SO}(\mathrm{Im}\,\mathbb{O}) \cong \mathrm{SO}(7)$. The induced action of G_2 on the Grassmannian of associative 3-dimensional subspaces in $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$ is transitive [17, p. 114], and the stabilizer of $\xi = \mathrm{Im}\,\mathbb{H}$ is $\mathrm{SO}(4) = \mathrm{Sp}(1) \times_{\mathbb{Z}_2} \mathrm{Sp}(1)$ so that $G_2/\mathrm{SO}(4)$ is the set of all associative 3-dimensional subspaces in $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$. By Lemma 2.1, there is an embedding of $G_2/\mathrm{SO}(4)$ in the Grassmannian $\widetilde{G}_3(\mathbb{R}^7)$ of *oriented* 3-dimensional subspaces of $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$. The action of $\mathrm{SO}(4)$ on $\mathrm{Im}\,\mathbb{O}$ is given by [17, p. 115]

$$g(a,b) = (q_1 \cdot a \cdot \overline{q}_1,\, q_2 \cdot b \cdot \overline{q}_1), \tag{2.3}$$

where we write $(a,b) = a + b \cdot e \in \mathbb{O}$ for $a, b \in \mathbb{H}$, and $g = [q_1, q_2]$ for some unit quaternions $q_1, q_2 \in \mathrm{Sp}(1)$. Thus $\xi^\perp \cong \mathbb{R}^4$ and $\xi \cong \Lambda^2_+$ as $\mathrm{SO}(4)$-representations, where Λ^2_- is the 3-dimensional representation of $\mathrm{SO}(4)$ on anti-self-dual 2-forms, i.e., [23, p. 155]

$$\mathrm{Im}\,\mathbb{O} = \xi \oplus \xi^\perp = \Lambda^2_- \oplus \mathbb{R}^4$$

as representations of $\mathrm{SO}(4)$.

Let J be the orthogonal complex structure (i.e., almost Hermitian structure) on $\varphi^\perp$ with $(1,0)$-space W and choose an orthonormal basis $e_1, \dots, e_4$ of $\varphi^\perp$ with $Je_1 = e_2$ and $Je_3 = e_4$. We say that J, or the corresponding W, is *positive* (resp. *negative*) according as the basis is positive (resp. negative).

On the other hand, following [10], we call a complex 2-dimensional subspace W *complex-coassociative* if $v \times w = 0$ for all $v, w \in W$. These notions are linked as follows.

Lemma 2.2. *Let W be a maximally isotropic subspace of $\varphi^\perp \otimes \mathbb{C}$. Then W is positive if and only if it is complex-coassociative.*

Proof. Setting $v = e_1 - \mathrm{i}e_2$ and $w = e_3 - \mathrm{i}e_4$, a simple calculation gives

$$(v \times w\,,\, \overline{v \times w}) = 4 - 2(e_1\,,\,[e_2, e_3, e_4]),$$

where $[\cdot,\cdot,\cdot]$ is the associator defined by $[u,v,w] = (u \cdot v) \cdot w - u \cdot (v \cdot w)$ [17, p. 114]. Now, by [17, Theorem 1.16], $e_1, \dots, e_4$ is a positive basis for $\varphi^\perp$ if and only if $(e_1, [e_2, e_3, e_4]) = 2$; the result follows. □

Lemma 2.3. *G_2 is transitive on pairs (u,v) of isotropic, orthogonal unit vectors with $u \times v = 0$.*

Proof. Let $u = e_1 - \mathrm{i}e_2$ and $v = e_3 - \mathrm{i}e_4$, with $e_1, \dots, e_4 \in \mathbb{R}^7$ orthogonal unit vectors. Then $u \times v = 0$ is equivalent to

$$e_1 \times e_3 - e_2 \times e_4 = 0 \quad \text{and} \quad e_1 \times e_4 + e_2 \times e_3 = 0,$$

from which it follows that $(e_1 \times e_2, e_3) = 0$. It is well known that G_2 is transitive on such ordered triples e_1, e_2, e_3, see [17, page 115]. Furthermore, $e_2 \times (e_1 \times e_3) = e_2 \times (e_2 \times e_4) = -e_4$, so that e_4 is determined by e_1, e_2, e_3; the result follows. □

2.2. **The standard representation of G_2 and the vector product on $\mathbb{R}^7$.** The most transparent way of describing the representation of G_2 on $\mathbb{R}^7$ is by describing its weight spaces, see [16, §22.3].

The Lie algebra $\mathfrak{g}_2$ has simple roots α_1 and α_2, and the remaining roots are given by

$$\alpha_3 = \alpha_1 + \alpha_2\,, \quad \alpha_4 = 2\alpha_1 + \alpha_2\,, \quad \alpha_5 = 3\alpha_1 + \alpha_2\,, \quad \alpha_6 = 3\alpha_1 + 2\alpha_2$$

and their negatives. The root lattice equals the weight lattice; the maximal root (i.e., the highest weight for the adjoint representation) is $3\alpha_1 + 2\alpha_2$. The representation $\mathrm{Im}\,\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^7$ of $\mathfrak{g}_2^{\mathbb{C}} = \mathfrak{g}_2 \otimes \mathbb{C}$ has weights $\pm\alpha_1$, $\pm(\alpha_1 + \alpha_2)$, $\pm(2\alpha_1 + \alpha_2)$ and 0.

For each weight λ, the corresponding weight space ℓ_λ is one-dimensional with $\overline{\ell}_\lambda = \ell_{-\lambda}$, and is isotropic for $\lambda \neq 0$. Clearly we have

$$\ell_\lambda \times \ell_\eta \subset \ell_{\lambda+\eta}\,. \tag{2.4}$$

For any subspace β of $\mathrm{Im}\,\mathbb{O}$, the *annihilator of* β is the subspace $\beta^a = \{L \in \mathrm{Im}\,\mathbb{O} : L \times \beta = 0\}$. It is easy to check [10] that, if β is isotropic of dimension one, then β^a is isotropic of dimension

three, and contains β; it suffices to calculate it when β is a weight space. We can prove the following in a similar fashion.

Lemma 2.4. *Let φ be an associative 3-dimensional subspace of* $\operatorname{Im}\mathbb{O}$.

(a) *Let β be a 1-dimensional isotropic subspace of $\varphi^\perp$. Then*

(i) *the unique positive (equivalently, complex-coassociative) maximally isotropic subspace of $\varphi^\perp$ which contains β is given by $W = \beta^a \cap \varphi^\perp$;*

(ii) *the unique negative (equivalently, not complex-coassociative) maximally isotropic subspace of $\varphi^\perp$ which contains β is given by $W = \overline{\beta}^\perp \cap \overline{\beta}^a \cap \varphi^\perp$.*

(b) *Let β be a 2-dimensional positive isotropic subspace of $\varphi^\perp$. Then $\beta = \beta^a \cap \varphi^\perp$.* □

2.3. **The twistor spaces of $G_2/\mathrm{SO}(4)$.** Let N be a Riemannian manifold. By a *twistor fibration of N (for harmonic maps)* is meant [6] an almost complex manifold (Z, J) (called a *twistor space*) and a fibration $\pi : Z \to N$ such that, for every (almost-)holomorphic map from a surface $\psi : M \to Z$, the composition $\varphi = \pi \circ \psi : M \to N$ is harmonic. Then φ is called the *twistor projection of ψ*, and ψ is called *a twistor lift of φ*.

For inner symmetric spaces, a general theory of such twistor fibrations is given in [6]. The twistor spaces are flag manifolds G/H where H is a centralizer of a torus.

Now $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and a maximal torus is given by $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$. Following (2.3), there are two more possibilities for centralizers of tori in $\mathfrak{so}(4)$:

$$\mathfrak{u}(2)_+ = \mathfrak{su}(2) \oplus \mathfrak{u}(1), \quad \mathfrak{u}(2)_- = \mathfrak{u}(1) \oplus \mathfrak{su}(2).$$

Hence there are precisely three flag manifold of G_2 fibring canonically over $G_2/\mathrm{SO}(4)$, which we shall denote by T_s $(s = 1, 2, 3)$ as in the following commutative diagram from [23, §11.9]; the fibration π_6 will be explained in §5.

$$\begin{tikzcd}
& & Z_3 = G_2/\mathrm{U}(1) \times \mathrm{U}(1) \arrow[dll, "\pi_4"'] \arrow[dd, "\pi_3"] \arrow[dr, "\pi_5"] & & \\
Z_1 = G_2/\mathrm{U}(2)_+ \arrow[drr, "\pi_1"'] & & & Z_2 = G_2/\mathrm{U}(2)_- \cong Q^5 \arrow[dl, "\pi_2"] \arrow[dr, "\pi_6"] & \\
& & G_2/\mathrm{SO}(4) & & S^6
\end{tikzcd} \tag{2.5}$$

Note that every fibre, apart from that of π_3, is isomorphic to $\mathbb{C}P^1$. The π_i $(i = 1, 2, 3)$ are fibre bundles over $G_2/\mathrm{SO}(4)$ associated to the principal G_2-bundle; more specifically:

$T_1 = G_2/\mathrm{U}(2)_+$ consists of all rank 2 isotropic subspaces of $\operatorname{Im}\mathbb{O} \otimes \mathbb{C} \cong \mathbb{C}^7$ which are complex-coassociative. The projection $\pi_1 : G_2/\mathrm{U}(2)_+ \to G_2/\mathrm{SO}(4)$ is given by $W \mapsto \{W \oplus \overline{W}\}^\perp$. By Lemma 2.2, the fibre at $\varphi \in G_2/\mathrm{SO}(4)$ is all positive maximally isotropic subspaces of $\varphi^\perp \otimes \mathbb{C}$, equivalently, all positive orthogonal complex structures on $\varphi^\perp$; this is the quaternionic twistor space of $G_2/\mathrm{SO}(4)$ [22].

$T_2 = Q^5$, the complex quadric $\big\{[z_0, \dots, z_5] \in \mathbb{C}P^6 : {z_0}^2 + \dots + {z_5}^2 = 0\big\}$ consisting of all isotropic lines in $\operatorname{Im}\mathbb{O} \otimes \mathbb{C}$. The projection $\pi_2 : Q^5 \to G_2/\mathrm{SO}(4)$ is given by $\pi_2(\ell) = \ell \oplus \overline{\ell} \oplus (\ell \times \overline{\ell})$.

Alternatively, we can think of T_2 as $G_2/\mathrm{U}(2)_-$, the space of all rank 2 isotropic subspaces of $\operatorname{Im}\mathbb{O} \otimes \mathbb{C}$ which are *not* complex-coassociative. There is a G_2-equivariant bundle isomorphism from Q^5 to $G_2/\mathrm{U}(2)_-$ given by $\ell \mapsto \ell^a \ominus \ell$ with inverse $W \mapsto W \times W$. With this model, the projection $\pi_2 : G_2/\mathrm{U}(2)_- \to G_2/\mathrm{SO}(4)$ is given by $W \mapsto \{W \oplus \overline{W}\}^\perp$. By Lemma 2.2, the fibre at $\varphi \in G_2/\mathrm{SO}(4)$ is all negative maximally isotropic subspaces of $\varphi^\perp \otimes \mathbb{C}$, equivalently, all negative orthogonal complex structures on $\varphi^\perp$.

$T_3 = G_2/(\mathrm{U}(1) \times \mathrm{U}(1))$ consists of all pairs (ℓ, D) where ℓ and D are subspaces of $\operatorname{Im}\mathbb{O} \otimes \mathbb{C}$ of ranks 1 and 2, respectively, which satisfy $\ell \subset D \subset \ell^a$. The projection $\pi_3 : G_2/(\mathrm{U}(1) \times \mathrm{U}(1)) \to G_2/\mathrm{SO}(4)$ is given by $\pi_3(\ell, D) = q \oplus \overline{q} \oplus (q \times \overline{q})$ where $q = D \ominus \ell$.

π_4 (resp. π_5): $G_2/(\mathrm{U}(1)\times\mathrm{U}(1)) \to G_2/\mathrm{U}(2)_\pm$ is given by $(\ell, D) \mapsto$ the positive (resp. negative) maximally isotropic subspace of $\varphi^\perp \times \mathbb{C}$ which contains ℓ, equivalently $\pi_5 : G_2/(\mathrm{U}(1)\times\mathrm{U}(1)) \to Q^5$ is given by $(\ell, D) \mapsto D \ominus \ell$.

For each twistor space, the theory of [6] gives a decomposition of the tangent bundle into horizontal and vertical parts and canonical almost complex structures J_1 and J_2 which agree on the horizontal space. The following description follows from the Lie theory in [6, Chapter 4] but will be explained in a more geometrical way in Section 4.4. We shall phrase it for smooth maps from a surface M, these are given by varying subspaces W, ℓ and D; we can regard these as subbundles of the trivial bundle $M \times (\mathrm{Im}\,\mathbb{O} \otimes \mathbb{C}) \cong M \times \mathbb{C}^7$ which we shall denote by $\underline{\mathbb{C}}^7$.

Lemma 2.5. *Let $\psi : M \to T_s$ be a smooth map from a surface to the twistor space T_s, and set $\varphi = \pi_s \circ \psi$. Then ψ is J_2-holomorphic (so that φ is harmonic) if and only if*

(s=1) *$\psi = W$ is a holomorphic subbundle of $\underline{\mathbb{C}}^7$ lying in $\ker A^\varphi_z$;*
(s=2) *$\psi = \ell$ is a holomorphic subbundle of $\underline{\mathbb{C}}^7$ lying in $\ker A^\varphi_z$;*
(s=3) *$\psi = (\ell, D)$ where ℓ and D are holomorphic subbundles of $\underline{\mathbb{C}}^7$ with $\ell \subset \ker A^\varphi_z$ and $A^\varphi_z(D) \subset \ell$.*

The same result holds for J_1-holomorphic if we replace 'holomorphic' by 'antiholomorphic' throughout.

Note that the maps in diagram (2.5) preserve horizontal spaces, but do not preserve J_1 or J_2.

3. Harmonic maps and their twistor lifts

3.1. **Harmonic maps into a Lie group.** Throughout the paper, all manifolds, bundles, and structures on them are taken to be C^∞-smooth. Recall that harmonic maps from surfaces enjoy conformal invariance (see, for example, [29]) so that the concept of harmonic map from a surface M is well defined. Let G be a compact Lie group. For any smooth map $\varphi : M \to G$, set $A^\varphi = \frac{1}{2}\varphi^{-1}\mathrm{d}\varphi$; thus A^φ is a 1-form with values in the Lie algebra $\mathfrak{g}$ of G; it is half the pull-back of the Maurer–Cartan form of G.

To study maps into a symmetric space G/H we embed G/H in the Lie group G by the totally geodesic Cartan embedding (see, for example, [9, Proposition 3.42]); this preserves harmonicity.

Further, any compact Lie group can be embedded totally geodesically in the unitary group $\mathrm{U}(n)$, so we now consider that group together with its standard action on $\mathbb{C}^n$. Let $\underline{\mathbb{C}}^n$ denote the trivial complex bundle $\underline{\mathbb{C}}^n = M \times \mathbb{C}^n$, then $D^\varphi = \mathrm{d} + A^\varphi$ defines a unitary connection on $\underline{\mathbb{C}}^n$. Let z be a local complex coordinate on an open set U of M and write $\mathrm{d}\varphi = \varphi_z \mathrm{d}z + \varphi_{\bar z}\mathrm{d}\bar z$, $A = A^\varphi_z \mathrm{d}z + A^\varphi_{\bar z}\mathrm{d}\bar z$, $D^\varphi = D^\varphi_z\mathrm{d}z + D^\varphi_{\bar z}\mathrm{d}\bar z$, $\partial_z = \partial/\partial z$ and $\partial_{\bar z} = \partial/\partial\bar z$. Then

$$A^\varphi_z = \frac{1}{2}\varphi^{-1}\varphi_z\,, \quad A^\varphi_{\bar z} = \frac{1}{2}\varphi^{-1}\varphi_{\bar z}\,, \quad D^\varphi_z = \partial_z + A^\varphi_z\,, \quad D^\varphi_{\bar z} = \partial_{\bar z} + A^\varphi_{\bar z}\,. \tag{3.1}$$

Interpreting A^φ_z and $A^\varphi_{\bar z}$ as endomorphisms, the adjoint of A^φ_z (with respect to the Hermitian inner product) is $-A^\varphi_{\bar z}$; in particular, if φ is real, i.e., maps into $\mathrm{O}(n)$, then A^φ_z is skew-symmetric, i.e., for any $p \in M$,

$$(A^\varphi_z(u), v) = -(u, A^\varphi_z(v)) \qquad (u, v \in \{p\} \times \mathbb{C}^n)\,, \tag{3.2}$$

where $(\cdot,\cdot)$ denotes the standard symmetric bilinear inner product on $\mathbb{C}^n$. We have some simple properties, as follows.

Lemma 3.1. (i) *If β is a holomorphic subbundle of $(\underline{\mathbb{C}}^n, D^\varphi_{\bar z})$, so is its* polar $\overline{\beta}^\perp$.

(ii) *If β is an isotropic line subbundle of $\underline{\mathbb{C}}^n$, then $(D^\varphi_{\bar z}(\beta), \beta) = 0$. Similar rules hold for D^φ_z, A^φ_z and $A^\varphi_{\bar z}$.*

Proof. (i) This follows from $\big(D^\varphi_{\bar z}(\overline{\beta}^\perp), \beta\big) = \partial_{\bar z}(\overline{\beta}^\perp, \beta) - \big(\overline{\beta}^\perp, D^\varphi_{\bar z}(\beta)\big) = 0$.

(ii) Similarly, $(D^\varphi_{\bar z}(\beta), \beta) = (1/2)\,\partial_{\bar z}(\beta, \beta) = 0$. The rest is proved in the same way. □

There is a unique holomorphic structure on $\underline{\mathbb{C}}^n$ with $\bar\partial$-operator given on each coordinate domain (U,z) by $D^\varphi_{\bar z}$ [19], we call this the *(Koszul–Malgrange) holomorphic structure induced by* φ; the resulting holomorphic vector bundle will be denoted by $(\underline{\mathbb{C}}^n, D^\varphi_{\bar z})$. Then [27] *a smooth map* $\varphi : M \to \mathrm{U}(n)$ *is harmonic if and only if, on each coordinate domain,* A^φ_z *is a holomorphic endomorphism of the holomorphic vector bundle* $(\underline{\mathbb{C}}^n, D^\varphi_{\bar z})$.

Now, for any holomorphic (or antiholomorphic) endomorphism E, at points where it does not have maximal rank, we shall 'fill out zeros' as in [8, Proposition 2.2] (cf. [25, §3.1]) to make its image and kernel into holomorphic subbundles $\mathrm{Im}\, E$ (which we often denote simply by $E(\underline{\mathbb{C}}^n)$) and $\ker E$ of $\underline{\mathbb{C}}^n$.

For any k,n with $0 \le k \le n$, let $G_k(\mathbb{C}^n)$ (resp. $G_k(\mathbb{R}^n)$) denote the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$ (resp. $\mathbb{R}^n$). The class of maps which we consider is given by the following generalization of a notion that F. E. Burstall [5] defined for harmonic maps $M \to G_k(\mathbb{C}^n)$.

Definition 3.2. A harmonic map $\varphi : M \to \mathrm{U}(n)$ from a surface is called *nilconformal* if $(A^\varphi_z)^r = 0$ for some $r \in \{1,2,\ldots\}$. We shall call the least such r the *nilorder of* φ *(in* $\mathrm{U}(n)$*)*.

Note that our concept of nilconformal reduces to that in [5] for maps into a complex Grassmannian embedded in $\mathrm{U}(n)$ via the Cartan embedding, but our *nilorder* may differ by ± 1 from that in [5]. *Harmonic maps of finite uniton number are nilconformal* (see [26]), but the converse is false, see Example 5.11 below.

For a non-constant nilconformal harmonic map $\varphi : M \to G_k(\mathbb{R}^n)$ or $G_k(\mathbb{C}^n)$, let $s(\varphi)$ denote the least positive integer s such $(A^\varphi_z)^{2s}\big((-1)^{s-1}\varphi\big) = 0$, where $-\varphi$ denotes $\varphi^\perp$. In general, $r - 1 \le 2s(\varphi) \le r+1$ where r is the nilorder of φ.

To study harmonic maps into $G_2/\mathrm{SO}(4)$, we embed $G_2/\mathrm{SO}(4)$ in G_2 by the Cartan embedding, and then G_2 into $\mathrm{SO}(7)$ and finally into $\mathrm{U}(7)$. Since the Cartan embedding of $G_2/\mathrm{SO}(4)$ in G_2 is the restriction of that of $G_3(\mathbb{R}^7)$ in $\mathrm{SO}(n)$ or of $G_3(\mathbb{C}^7)$ in $\mathrm{U}(n)$, we get the same result by embedding $G_2/\mathrm{SO}(4)$ in the real Grassmannian $G_3(\mathbb{R}^7)$, then into the complex Grassmannian $G_3(\mathbb{C}^7)$, and finally via the Cartan embedding into $\mathrm{U}(7)$. In particular, a harmonic map $M \to G_2/\mathrm{SO}(4)$ is nilconformal if it is nilconformal as a map into $\mathrm{U}(7)$. We shall repeatedly use the following simple rules: we give (iii) in the form that we need it, but it is true in a much more general context.

Lemma 3.3. (i) *Let* $v,w : M \to \mathbb{C}^7$ *Then* $A^\varphi_z(v \times w) = A^\varphi_z(v) \times w + v \times A^\varphi_z(w)$. *Similar rules hold for* $A^\varphi_{\bar z}$, D^φ_z *and* $D^\varphi_{\bar z}$.

(ii) *If* β *is a holomorphic subbundle of* $(\underline{\mathbb{C}}^7, D^\varphi_{\bar z})$, *so is its annihilator* β^a.

(iii) *If* $\varphi : M \to G_2/\mathrm{SO}(4)$ *is a harmonic map, and* β *is a holomorphic line subbundle of* $\varphi^\perp$, *then the subbundles* W *of* $\varphi^\perp$ *given by the formulae of Lemma 2.4(a) are holomorphic.*

Proof. (i) and (ii) Since φ has values in G_2, A^φ_z has values in $\mathfrak{g}_2 \otimes \mathbb{C}$. The other parts follow.

(iii) This follows from Lemma 3.1(ii) and $(D^\varphi_{\bar z} W, \beta) = (W, D^\varphi_z \beta) = 0$. $\square$

3.2. **Construction of the twistor lifts.** We may now state one half of our main result, that nilconformal harmonic maps have twistor lifts; in fact, we shall give explicit formulae for those lifts. The converse will follow from Proposition 4.7, from an interpretation using geometric flag manifolds.

Theorem 3.4. *Let* $\varphi : M \to G_2/\mathrm{SO}(4)$ *be a non-constant nilconformal harmonic map. Then* $s(\varphi) \in \{1,2,3\}$, *and* φ *has a* J_2*-holomorphic lift from* M *to the twistor space* T_s *where* $s = s(\varphi)$.

Proof. Suppose that $\varphi : M \to G_2/\mathrm{SO}(4)$ is a non-constant nilconformal harmonic map. We first note that, by nilpotency, $\mathrm{rank}(A^\varphi_z)^{2i}(\varphi) \le 3 - i \ \ (i = 1,2,\ldots)$, in particular, $s(\varphi) \le 3$.

We now construct a J_2-holomorphic lift into T_s where $s = s(\varphi)$. First, since A^φ_z is holomorphic, we may fill out zeros as described above to obtain a holomorphic subbundle $\beta = (A^\varphi_z)^s((-1)^{s-1}\varphi)$ (i.e., $\beta = \mathrm{Im}\{(A^\varphi_z)^s|_{(-1)^{s-1}\varphi}\}$) of $\varphi^\perp$. Then $(\beta, \beta) = \big((A^\varphi_z)^{2s}((-1)^{s-1}\varphi), (-1)^{s-1}\varphi\big)$ which is 0 by definition of $s(\varphi)$; hence β *is an isotropic holomorphic subbundle of* $\varphi^\perp$, which is thus of rank 0, 1 or 2. Now let W be a maximally isotropic holomorphic subbundle of $\varphi^\perp$ containing β, then

$$(A^\varphi_z)^s((-1)^{s-1}\varphi) = \beta \subset W \subset \overline{\beta}^\perp = \ker(A^\varphi_z)^s|_{\varphi^\perp}\,, \tag{3.3}$$

the last equality following from (3.2). From (3.3) and the definition of $s(\varphi)$ we see that

$$(A^\varphi_z)^i(W) = 0 \quad \text{if and only if} \quad i \geq s(\varphi). \tag{3.4}$$

When $s = 3$, we will choose W to satisfy the additional condition

$$(A^\varphi_z)^2(W) \subset W \tag{3.5}$$

(which is automatic from (3.4) for $s = 1, 2$). We build the lift of φ from such a W and its derivatives $(A^\varphi_z)^i(W)$ $(i \leq s-1)$, note that *these are all holomorphic subbundles of φ or $\varphi^\perp$, and they are all isotropic since* $\big(A^\varphi_z(W), A^\varphi_z(W)\big) = \big((A^\varphi_z)^2(W), W\big) = 0$ by (3.5); further, *the last non-zero one,* $(A^\varphi_z)^{s-1}(W)$*, is in* $\ker A^\varphi_z$. To get a lift into T_s, we will choose W to be positive when s is odd and negative when s is even. We give the details and explicit formulae for the three possible values of $s(\varphi)$ in the next three lemmas.

Lemma 3.5. *Let $\varphi : M \to G_2/\mathrm{SO}(4)$ be a nilconformal harmonic map with $s(\varphi) = 1$. Then there is a J_2-holomorphic lift $W : M \to T_1 = G_2/\mathrm{U}(2)_+$ of φ given by*

$$W = \beta^a \cap \varphi^\perp \quad \textit{where} \quad \beta = A^\varphi_z(\varphi). \tag{3.6}$$

Proof. First note that $\beta = 0$ would imply that φ is constant, so that β has rank 1 or 2.

(a) If $\operatorname{rank}\beta = 1$, then by Lemma 2.4, β has a unique extension to a positive, equivalently complex-coassociative, maximally isotropic subbundle W given by (3.6) and this is holomorphic by Lemma 3.3.

(b) If $\operatorname{rank}\beta = 2$, set $W = \beta$. We claim that W is complex-coassociative. To see this, let $v, w \in \Gamma(\varphi)$ — here $\Gamma(\cdot)$ denotes the space of (smooth) sections of a vector bundle. Since φ is associative, $v \times w \in \Gamma(\varphi)$. Applying A^φ_z, by Lemma 3.3(i), we have that $A^\varphi_z(v) \times w + v \times A^\varphi_z(w)$ lies in $A^\varphi_z(\varphi)$. Applying A^φ_z again, since by $s(\varphi) = 1$ we have $(A^\varphi_z)^2(\varphi) = 0$, we obtain $A^\varphi_z(v) \times A^\varphi_z(w) = 0$. Hence W is complex-coassociative.

Since, by Lemma 2.4(b), $\beta^a \cap \varphi^\perp = \beta$ when β is complex-coassociative of rank 2; the lemma follows. $\square$

Lemma 3.6. *Let $\varphi : M \to G_2/\mathrm{SO}(4)$ be a nilconformal harmonic map with $s(\varphi) = 2$. Then there is a J_2-holomorphic lift $\ell : M \to T_2 = Q^5$ of φ given by*

$$\ell = A^\varphi_z(W) \quad \textit{where} \quad W = \overline{\beta}^\perp \cap \overline{\beta}^a \cap \varphi^\perp \ \textit{with}\ \beta = (A^\varphi_z)^2(\varphi^\perp). \tag{3.7}$$

Proof. (a) Suppose first that $\operatorname{rank}\beta = 0$, then set $\ell = A^\varphi_z(\varphi^\perp)$ Then ℓ is a non-zero holomorphic isotropic subbundle of φ, and is thus of rank 1; further, it is in the kernel of A^φ_z. It thus gives a J_2-holomorphic lift of φ into Q^5. Note that it is given by (3.7) with $\beta = 0$ since the middle formula gives $W = \varphi^\perp$. (In fact, it can be checked that taking W to be *any* maximally isotropic subbundle in $\varphi^\perp$ gives the same ℓ.)

(b) Next suppose that $\operatorname{rank}\beta \neq 0$. If $\operatorname{rank}\beta = 1$, by Lemma 2.4, it has a unique extension to a maximally isotropic subbundle W which is negative, equivalently, not complex-coassociative, and this is holomorphic by Lemma 3.3. If $\operatorname{rank}\beta = 2$, set $W = \beta$ (which may or may not be complex-coassociative).

Whether $\operatorname{rank}\beta = 1$ or 2, set $\ell = A^\varphi_z(W)$, as explained already, gives a J_2-holomorphic lift of φ into Q^5.

Further, with notation which will be convenient in §4.4, set $\psi_2 = \ell$, $\psi_{-2} = \overline{\psi}_2$ and $\psi_0 = \psi_{-2} \times \psi_2$, then $\varphi = \psi_{-2} \oplus \psi_0 \oplus \psi_2$ and, by Lemma 2.1(v), $\psi_0 \times \psi_2 = \psi_2$. Now $\big((A^\varphi_z)^2(\varphi), \psi_2\big) = \big((A^\varphi_z)^2(\varphi), A^\varphi_z(W)\big) = \big(\varphi, (A^\varphi_z)^3(W)\big) = 0$ so that $(A^\varphi_z)^2(\varphi)$ lies in the polar of ψ_2, i.e.,

$$(A^\varphi_z)^2(\varphi) \subset \psi_0 \oplus \psi_2\,. \tag{3.8}$$

Similarly, $(A^\varphi_z(\varphi^\perp), \psi_2) = (\varphi^\perp, A^\varphi_z(\psi_2)) = 0$, so that $A^\varphi_z(\varphi^\perp) \subset \psi_0 \oplus \psi_2$. Hence $\beta = (A^\varphi_z)^2(\varphi^\perp) \subset A^\varphi_z(\psi_0)$ and since ψ_0 has rank one, so has β, so that an explicit formula for W is given by Lemma 2.4(a)(ii). The lemma follows.

Further, applying A^φ_z to $\psi_0 \times \psi_2 \subset \psi_2$ gives $A^\varphi_z(\psi_0) \times \psi_2 = 0$, so $\beta \times \ell = 0$, so $\beta \subset \ell^a \cap \varphi^\perp$. It follows that when $\operatorname{rank}\beta = 1$, $W = \ell^a \cap \varphi^\perp$ as this contains β and is negative. When $\operatorname{rank}\beta = 0$, we can take W to be $\ell^a \cap \varphi^\perp$, then, in all cases, $W \times W = \ell$. Thus W gives the lift into T_2 thought of as $G_2/\mathrm{U}(2)_-$, see also §4.4. □

Lemma 3.7. *Let $\varphi : M \to G_2/\mathrm{SO}(4)$ be a nilconformal harmonic map with $s(\varphi) = 3$. Then there is a J_2-holomorphic lift $(\ell, D) : M \to T_3 = G_2/(\mathrm{U}(1) \times \mathrm{U}(1))$ of φ given by*

$$(3.9)\quad \ell = (A^\varphi_z)^2(W),\ D = \operatorname{span}\{A^\varphi_z(W), (A^\varphi_z)^2(W)\} \quad \textit{where} \quad W = \beta^a \cap \varphi^\perp \ \textit{with}\ \beta = (A^\varphi_z)^3(\varphi).$$

Proof. (i) We first show that there is a maximally isotropic holomorphic subbundle $\widetilde{W}$ of $\varphi^\perp$ which contains β and satisfies (3.5).

Suppose that $\operatorname{rank}\beta = 0$. Then $(A^\varphi_z)^4(\varphi^\perp) = 0$ so that $s(\varphi) \le 2$, a contradiction.

Suppose, next, that $\operatorname{rank}\beta = 2$, then set $\widetilde{W} = \beta$ — this clearly satisfies (3.5).

Finally suppose that $\operatorname{rank}\beta = 1$. Since β is closed under $(A^\varphi_z)^2$, so is its polar $\overline{\beta}^\perp$ so that $(A^\varphi_z)^2$ factors to an endomorphism B of the rank 2 quotient bundle $\overline{\beta}^\perp/\beta$. Then, *either* $B = 0$, in which case we may define $\widetilde{W}$ to be $\beta + \gamma$ where γ is either of the two isotropic line subbundles in $\overline{\beta}^\perp/\beta$ — explicit formulae are given by 2.4(a)(i) or (ii) — *or* $B^2 = 0$, in which case we set $\widetilde{W} = \beta + (A^\varphi_z)^2(\overline{\beta}^\perp)$. In either case, $\widetilde{W}$ satisfies (3.5); also $A^\varphi_z(\widetilde{W})$ and $(A^\varphi_z)^2(\widetilde{W})$ are isotropic, and non-zero by (3.4), and so both have rank one.

(ii) Using part (i) we now show that we can actually choose a W which is positive, equivalently, complex-coassociative. There are two cases, as follows.

(a) Suppose that $A^\varphi_z(\beta) = 0$. Then $\operatorname{rank}\beta = 1$, otherwise $s(\varphi) = 1$ by (3.4). As in Lemma 2.4(a)(i), set $W = \beta^a \cap \varphi^\perp$. Then W is a positive holomorphic maximally isotropic subbundle of $\varphi^\perp$ which contains β. Further, $W \times \beta = 0$. Applying A^φ_z twice gives $(A^\varphi_z)^2(W) \times \beta = 0$, showing that W satisfies (3.5).

(b) Suppose, instead, that $A^\varphi_z(\beta) \neq 0$. Let $\widetilde{W}$ be a maximally isotropic subbundle of $\varphi^\perp$ of either sign containing β and satisfying (3.5) as constructed by part (i). Since $A^\varphi_z(\beta)$ is contained in $A^\varphi_z(\widetilde{W})$ and they both have rank one, they must be equal, whence $(A^\varphi_z)^2(\beta) = (A^\varphi_z)^2(\widetilde{W})$, which is non-zero by (3.4). This implies that $\operatorname{rank}\beta = 2$; so set $W = \beta$; this satisfies (3.5). To see that this W is positive, as for the case $s = 2$, we decompose $\varphi = \psi_{-2} \oplus \psi_0 \oplus \psi_2$ where $\psi_2 = A^\varphi_z(W)$, $\psi_{-2} = \overline{\psi}_2$ and $\psi_0 = \psi_{-2} \times \psi_2$ and again (3.8) holds. Let $\Psi \in \Gamma(\varphi)$ and $\Psi_2 \in \Gamma(\psi_2)$. Then from (3.8) and associativity of φ (see Lemma 2.1(v)), $(A^\varphi_z)^2(\Psi) \times \Psi_2 \in \Gamma(\psi_2)$. Applying A^φ_z, we see that $(A^\varphi_z)^3(\Psi) \times \Psi_2 + (A^\varphi_z)^2(\Psi) \times A^\varphi_z(\Psi_2)$ lies in $A^\varphi_z(\psi_2)$. Now $(A^\varphi_z)^2(\psi_2) = 0$ and $(A^\varphi_z)^4(\psi_2) = 0$ so applying A^φ_z once more gives then $(A^\varphi_z)^3(\Psi) \times A^\varphi_z(\Psi_2) = 0$, i.e., $W \times (A^\varphi_z)^2(W) = 0$, so that W is complex-coassociative, i.e., positive.

By Lemma 2.4(b), $\beta^a \cap \varphi^\perp = \beta$ when β is a positive and of rank 2, thus, *in both cases (a) and (b), $W = \beta^a \cap \varphi^\perp$ provides a positive maximally isotropic subbundle of $\varphi^\perp$ containing β and satisfying* (3.5).

We set $\ell = (A^\varphi_z)^2(W)$. Since $A^\varphi_z(W)$ and $(A^\varphi_z)^2(W)$ are in φ and $\varphi^\perp$, respectively, and so linearly independent, they span a rank 2 subbundle D; by (3.5) this is isotropic. The subbundles ℓ and D are clearly holomorphic and satisfy $A^\varphi_z(D) \subset \ell$; by (3.4) ℓ is in the kernel of A^φ_z.

Further, W is complex-coassociative and satisfies (3.5), Applying A^φ_z gives $A^\varphi_z(W) \times (A^\varphi_z)^2(W) = 0$. It follows that $D \subset \ell^a$. The lemma follows. □

The three lemmas we have just established complete the proof of Theorem 3.4. □

4. Twistor spaces as flags

To understand our constructions better, we embed our exceptional symmetric space in a Grassmannian; we now recall some methods for those.

4.1. **Harmonic maps into real and complex Grassmannians.** For the case of real or complex Grassmannians, the twistor spaces constructed by [26] admit descriptions as geometric flag manifolds, as we now explain.

As before, for any k, n with $0 \leq k \leq n$, let $G_k(\mathbb{C}^n)$ denote the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$. We identify a smooth map $\varphi : M \to G_k(\mathbb{C}^n)$ with the rank k subbundle of $\underline{\mathbb{C}}^n = M \times \mathbb{C}^n$, also denoted by φ, whose fibre at $p \in M$ is $\varphi(p)$.

For a subbundle V of $\underline{\mathbb{C}}^n$ we denote by π_V (resp. $\pi_V^\perp$) orthogonal projection from $\underline{\mathbb{C}}^n$ to V (resp. to its orthogonal complement $V^\perp$). The Cartan embedding is given by

$$\iota : G_k(\mathbb{C}^n) \hookrightarrow \mathrm{U}(n), \quad \iota(V) = \pi_V - \pi_V^\perp ; \tag{4.1}$$

this is totally geodesic, and isometric up to a constant factor. We shall identify V with its image $\iota(V)$; since $\iota(V^\perp) = -\iota(V)$, this identifies $V^\perp$ with $-V$.

We consider the real Grassmannian $G_k(\mathbb{R}^n)$ to be the totally geodesic submanifold $\{V \in G_k(\mathbb{C}^n) : V = \overline{V}\}$.

We now recall some methods for studying maps into Grassmannians [8]. Any subbundle φ of $\underline{\mathbb{C}}^n$ inherits a metric by restriction, and a connection ∇_φ by orthogonal projection:

$$(\nabla_\varphi)_Z(v) = \pi_\varphi(\partial_Z v) \qquad (Z \in \Gamma(TM),\ v \in \Gamma(\varphi)\,).$$

Let φ and ψ be two mutually orthogonal subbundles of $\underline{\mathbb{C}}^n$. By the *∂' and ∂''-second fundamental forms of φ in $\varphi \oplus \psi$* we mean the vector bundle morphisms $A'_{\varphi,\psi}\,, A''_{\varphi,\psi} : \varphi \to \psi$ defined on each coordinate domain (U, z) by

$$A'_{\varphi,\psi}(v) = \pi_\psi(\partial_z v) \quad \text{and} \quad A''_{\varphi,\psi}(v) = \pi_\psi(\partial_{\bar z} v) \qquad (v \in \Gamma(\varphi)\,). \tag{4.2}$$

The following follow from the definition, the last from Lemma 3.1(ii).

Lemma 4.1. *For any mutually orthogonal subbundles φ and ψ of $\underline{\mathbb{C}}^n$,*

(i) *$A''_{\psi,\varphi}$ is minus the adjoint of $A'_{\varphi,\psi}$;*
(ii) *$A''_{\overline{\varphi},\overline{\psi}}$ is the conjugate of $A'_{\varphi,\psi}$;*
(iii) *if φ is isotropic and of rank one, $A'_{\varphi,\overline{\varphi}}$ and $A''_{\varphi,\overline{\varphi}}$ vanish.* □

In particular, we have the *second fundamental forms of* φ: $A'_\varphi = A'_{\varphi,\varphi^\perp} : \varphi \to \varphi^\perp$ and $A''_\varphi = A''_{\varphi,\varphi^\perp} : \varphi \to \varphi^\perp$; on identifying $\varphi : M \to G_k(\mathbb{C}^n)$ with its composition $\iota \circ \varphi : M \to \mathrm{U}(n)$ with the Cartan embedding, we see that the fundamental endomorphism A^φ_z of (3.1) is minus the direct sum of A'_φ and $A'_{\varphi^\perp}$, similarly the connection D^φ of the last section is the direct sum of ∇_φ and $\nabla_{\varphi^\perp}$. It follows that *a smooth map $\varphi : M \to G_k(\mathbb{C}^n)$ is harmonic if and only if A'_φ is holomorphic, i.e., $A'_\varphi \circ \nabla''_\varphi = \nabla''_{\varphi^\perp} \circ A'_\varphi$*, where we write $\nabla''_\varphi = (\nabla_\varphi)_{\partial/\partial\bar z}$, see also [8, Lemma 1.3].

We fill out zeros as in §3.1 to obtain subbundles $G'(\varphi) = \operatorname{Im} A'_\varphi = \operatorname{Im}(A^\varphi_z|_\varphi)$ and $G''(\varphi) = \operatorname{Im} A''_\varphi = \operatorname{Im}(A^\varphi_{\bar z}|_\varphi)$, called the *$\partial'$- and ∂''-Gauss transforms* or *Gauss bundles of* φ; these define maps into Grassmannians which are also harmonic, see [8, Proposition 2.3], [29]; in fact, these operations are examples of adding a uniton in the sense of K. Uhlenbeck [27].

More generally, we define the *ith ∂'-Gauss transform* $G^{(i)}(\varphi)$ of a harmonic map $\varphi : M \to G_k(\mathbb{C}^n)$ by $G^{(0)}(\varphi) = \varphi$, $G^{(i)}(\varphi) = G'(G^{(i-1)}(\varphi))$, and the *ith ∂''-Gauss transform* $G^{(-i)}(\varphi)$ $G^{(-i)}(\varphi) = G''(G^{(-i+1)}(\varphi))$. Note that $G^{(1)}(\varphi) = G'(\varphi)$ and $G^{(-1)}(\varphi) = G''(\varphi)$. The sequence $(G^{(i)}(\varphi))_{i\in\mathbb{Z}}$ of harmonic maps is called [28] the *harmonic sequence of* φ. By Lemma 4.1, if φ is a harmonic map into a *real* Grassmannian, we have $A''_\varphi = \overline{A'_\varphi}$ and $G^{(-i)}(\varphi) = \overline{G^{(i)}(\varphi)}$ $\forall\, i \in \mathbb{Z}$.

A harmonic map $\varphi : M \to G_k(\mathbb{C}^n)$ is called *strongly conformal* [8] if $G'(\varphi)$ and $G''(\varphi)$ are orthogonal, equivalently $s(\varphi) = 1$. For a harmonic map $\varphi : M \to G_k(\mathbb{R}^n)$ into a real Grassmannian, strong conformality is equivalent to isotropy of the subbundle $G'(\varphi)$.

4.2. **Twistor lifts of maps into complex Grassmannians.** For any complex vector spaces or vector bundles E, F, $\mathrm{Hom}(E, F) = \mathrm{Hom}_{\mathbb{C}}(E, F)$ will denote the vector space or bundle of (complex-)linear maps from E to F.

Let $n, t, d_0, d_1, \dots, d_t$ be positive integers with $\sum_{i=0}^t d_i = n$. Let $F = F_{d_0,\dots,d_t}$ be the (complex) flag manifold $\mathrm{U}(n)/\mathrm{U}(d_0) \times \cdots \times \mathrm{U}(d_t)$. The elements of F are $(t+1)$-tuples $\psi = (\psi_0, \psi_1, \dots, \psi_t)$ of mutually orthogonal subspaces with $\psi_0 \oplus \cdots \oplus \psi_t = \mathbb{C}^n$; we call these subspaces the *legs of* ψ. There is a canonical embedding of F into the product $G_{d_0}(\mathbb{C}^n) \times \cdots \times G_{d_0+\cdots+d_{t-1}}(\mathbb{C}^n)$ given by sending $(\psi_0, \psi_1, \dots, \psi_t)$ to its *associated flag* $(T_0, \dots, T_{t-1})$ where $T_i = \sum_{j=0}^i \psi_j$; the restriction to F of the Kähler structure on this product is an (integrable) complex structure which we denote by J_1.

Set $k = \sum_{j=0}^{[t/2]} d_{2j}$ and $N = G_k(\mathbb{C}^n)$. We define a mapping which combines the even-numbered legs:

$$\pi_e : F_{d_0,\dots,d_t} \to G_k(\mathbb{C}^n), \qquad \psi = (\psi_0, \psi_1, \dots, \psi_t) \mapsto \sum_{j=0}^{[t/2]} \psi_{2j}; \tag{4.3}$$

we could instead define a projection π_{odd} which combines odd-numbered legs, so that $\pi_{odd}(\psi)$ is the orthogonal complement of $\pi_e(\psi)$. The projection π_e is a Riemannian submersion with respect to the natural homogeneous metrics on F and $G_k(\mathbb{C}^n)$ so that each tangent space decomposes into the direct sum of the *vertical space*, made up of the tangents to the fibres, and its orthogonal complement, the *horizontal space*.

We define an almost complex structure J_2 by changing the sign of J_1 on the vertical space. This gives $(1,0)$-horizontal and vertical spaces as follows:

$$\mathcal{H}^{J_2}_{1,0} = \sum_{\substack{i,j=0,\dots,t \\ i<j,\ j-i \text{ odd}}} \mathrm{Hom}(\psi_i, \psi_j)\,, \qquad \mathcal{V}^{J_2}_{1,0} = \sum_{\substack{i,j=0,\dots,t \\ j<i,\ j-i \text{ even}}} \mathrm{Hom}(\psi_i, \psi_j).$$

The almost complex structure J_2 is not integrable except in the trivial case $t = 1$, see [7]. However,

Proposition 4.2. [6] *$\pi_e : (F, J_2) \to G_k(\mathbb{C}^n)$ is a twistor fibration for harmonic maps.*

Now let M be a surface, and let $\psi = (\psi_0, \psi_1, \dots, \psi_t) : M \to F$ be a smooth map. The last proposition says that if ψ is J_2-holomorphic, then its twistor projection $\pi_e \circ \psi$ is harmonic. From [5] we have:

Proposition 4.3. [26] *Let $\psi = (\psi_0, \psi_1, \dots, \psi_t) : M \to F$ be a smooth map. Then*

(i) *ψ is J_1-holomorphic if and only if $A'_{\psi_i,\psi_j} = 0$ whenever $i - j$ is positive*;

(ii) *ψ is J_2-holomorphic if and only if*

$$A'_{\psi_i,\psi_j} = 0 \text{ when } i-j \text{ is positive and odd, or } j-i \text{ is positive and even.} \quad \square \tag{4.4}$$

In [26], the authors developed a general method for producing twistor lifts of harmonic maps which used the following definition.

Definition 4.4. Let $\varphi : M \to \mathrm{U}(n)$ be a smooth map. A filtration of $\underline{\mathbb{C}}^n$ by subbundles:

$$\underline{\mathbb{C}}^n = Z_0 \supset Z_1 \supset \cdots \supset Z_t \supset Z_{t+1} = \underline{0} \tag{4.5}$$

is called an *A^φ_z-filtration (of length t)* if, for each $i = 0, 1, \dots, t$,

(i) Z_i is a holomorphic subbundle, i.e., $\Gamma(Z_i)$ is closed under $D^\varphi_{\bar z}$;

(ii) A^φ_z maps Z_i into the smaller subbundle Z_{i+1} .

Note that each Z_i is a *uniton* [27] for φ, i.e., a holomorphic subbundle of $(\underline{\mathbb{C}}^n, D^\varphi_{\bar z})$, which is closed under A^φ_z, and any uniton can be embedded in a A^φ_z-filtration [26].

Clearly, A^φ_z-filtrations of φ exist if and only if φ is nilconformal. By the *legs* of a filtration (4.5) we mean the subbundles $\psi_i = Z_i \ominus Z_{i+1}$ $(i = 0, \dots, t)$. We say that a filtration (4.5) is (i) *strict* if all the inclusions $Z_{i+1} \subset Z_i$ are strict, equivalently, all the legs ψ_i are non-zero; (ii) *alternating* (*for* $\pm\varphi$) if

$$\psi_i \subset (-1)^i \varphi \qquad \text{or} \quad \psi_i \subset (-1)^{i+1} \varphi \quad (i = 0, 1, \dots, t). \tag{4.6}$$

Let $\varphi : M \to \mathrm{U}(n)$ be nilconformal of nilorder r. Then two examples of A^φ_z-filtrations are the *filtration by A^φ_z-images* given by $Z_i = \mathrm{Im}(A^\varphi_z)^i$ and its 'dual', the *filtration by A^φ_z-kernels*: $Z_i = \ker(A^\varphi_z)^{r-i}$. By nilpotency, these filtrations are strict; they coincide when $r = n$, in which case their legs are all of rank one.

The point of A^φ_z-filtrations is the following result from [26].

Proposition 4.5. *Let $\varphi : M \to G_k(\mathbb{C}^n)$ be a nilconformal harmonic map from a surface. Then setting $\psi_i = Z_{i+s} \ominus Z_{i+s+1}$ defines a one-to-one correspondence between J_2-holomorphic lifts $\psi = (\psi_0, \psi_1, \dots, \psi_t)$ of φ into a complex flag manifold $F_{d_0,\dots,d_t}$ and strict alternating A^φ_z-filtrations* (4.5) *of length t.*

4.3. **Twistor lifts of maps into real Grassmannians.** Let $G_k(\mathbb{R}^n)$ denote the Grassmannian of real k-dimensional subspaces of $\mathbb{C}^n$, and $\mathfrak{p} : \widetilde{G}_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$ its oriented double cover where $\widetilde{G}_k(\mathbb{R}^n)$ consists of *oriented* real k-dimensional subspaces. Note that $\widetilde{G}_k(\mathbb{R}^n)$ is inner if and only if k or $n-k$ is even [6, p. 38]: the theory in that paper applies to this case. Since taking the orthogonal complement defines an isometry from $G_k(\mathbb{R}^n)$ to $G_{n-k}(\mathbb{R}^n)$, we shall assume, without loss of generality, that $n-k$ is even. To obtain twistor spaces and harmonic maps into these Grassmannians, we start with a filtration (4.5) which is *real* in the sense that $Z_i = \overline{Z}^\perp_{t+1-i}$, equivalently, $\psi_i = \overline{\psi_{t-i}}$ $(i = 0, 1, \dots, t)$. Then, if the filtration is strict and alternating and the length t of the filtration is odd, it has an even number of legs, and ψ projects under π_e (or π_{odd}) to a map into $\mathrm{SO}(2m)/\mathrm{U}(m)$, see [26].

We are more interested in the case when t is even. In that case, it is convenient to set $t = 2s$ and renumber the legs: $\psi = (\psi_{-s}, \dots, \psi_0, \dots, \psi_s)$ so that these satisfy the reality condition

$$\psi_{-i} = \overline{\psi_i} \qquad (i = 1, \dots, s)\,; \tag{4.7}$$

thus all the legs are determined by ψ_i with i positive, and the middle leg $\psi_0 = \underline{\mathbb{C}}^n \ominus \sum_{i=1}^s (\psi_i + \overline{\psi_i})$ is real.

This suggests that, for any natural numbers $d_0, \dots, d_s$ with $d_0 + 2(d_1 + \cdots + d_s) = n$, we define submanifolds of the complex flag manifolds in the last section by

$$F^{\mathbb{R}}_{d_s,\dots,d_0} = \left\{\psi = (\psi_{-s}, \dots, \psi_0, \dots, \psi_s) \in F_{d_{-s},\dots,d_0,\dots,d_s} : \psi_i = \overline{\psi}_{-i}\ \forall i\right\}. \tag{4.8}$$

Here we set $d_{-i} = d_i$ and the *order* of the indices is the same as in [26]. As a homogeneous space, $F^{\mathbb{R}}_{d_s,\dots,d_0} = \mathrm{O}(n)/H$, where $H = \mathrm{U}(d_s) \times \cdots \times \mathrm{U}(d_1) \times \mathrm{O}(d_0)$; equally well, $F^{\mathbb{R}}_{d_s,\dots,d_0} = \mathrm{SO}(n)/\widetilde{H}$ where $\widetilde{H} = \mathrm{U}(d_s) \times \cdots \times \mathrm{U}(d_1) \times \mathrm{SO}(d_0)$. The twistor projection π_e (or π_{odd}) of (4.3) restricts to give twistor fibrations: $\pi^{\mathbb{R}}_0 : F^{\mathbb{R}}_{d_s,\dots,d_0} \to \mathrm{O}(n)/\mathrm{O}(k) \times \mathrm{O}(n-k) = G_k(\mathbb{R}^n)$ where $n-k = 2\sum_{i \in \mathbb{N},\ i \text{ odd}} d_i$ and $\pi^{\mathbb{R}}_0(\psi) = \sum_{i \in \mathbb{Z},\ i \text{ even}} \psi_i$, and $\widetilde{\pi}^{\mathbb{R}}_0 : F^{\mathbb{R}}_{d_s,\dots,d_0} = \mathrm{SO}(n)/\widetilde{H} \to \mathrm{SO}(n)/\mathrm{SO}(k) \times \mathrm{SO}(n-k) = \widetilde{G}_k(\mathbb{R}^n)$, providing twistor spaces for $G_k(\mathbb{R}^n)$ and $\widetilde{G}_k(\mathbb{R}^n)$ when $n-k$ is even.

For a map $\varphi : M \to \widetilde{G}_k(\mathbb{R}^n)$ from a surface, we define A^φ_z by forgetting the orientation of φ, i.e., $A^\varphi_z = a^{\mathfrak{p}\circ\varphi}$. We have the following version of Proposition 4.5 adapted to the real case.

Proposition 4.6. *Let $\varphi : M \to \widetilde{G}_k(\mathbb{R}^n)$ be a nilconformal harmonic map where $n-k$ is even. Then setting $\psi_i = Z_{i+s} \ominus Z_{i+s+1}$ defines a one-to-one correspondence between J_2-holomorphic lifts $\psi = (\psi_{-s}, \dots, \psi_0, \dots, \psi_s)$ of φ into a flag manifold* (4.8) *and* real *strict alternating A^φ_z-filtrations* (4.5) *of length $2s$.*

Proposition 4.7. *Let $\psi : M \to F^{\mathbb{R}}_{d_s,\dots,d_0}$ be a J_2-holomorphic map into a flag manifold* (4.8)*. Suppose that its twistor projection $\varphi := \pi^{\mathbb{R}}_0 \circ \psi : M \to \widetilde{G}_k(\mathbb{R}^n)$ is non-constant. Then φ is a nilconformal harmonic map into an oriented real Grassmannian with $n-k$ even. Further $s(\varphi) \le s$.*

Proof. That φ is harmonic follows from Proposition 4.2. Since $(-1)^{s-1}\varphi = \sum_{i=0}^{s-1} \psi_{-s+1+2i}$ and $(A^\varphi_z)^2$ maps $\psi_{-s+1+2i}$ to $\sum_{j>i} \psi_{-s+1+2j}$, we see that $(A^\varphi_z)^{2s}\big((-1)^{s-1}\varphi\big) = 0$ so that $s(\varphi) \le s$. □

Remark 4.8. Remark 5.15 shows that a nilconformal harmonic map may have lifts into more than one twistor space; thus the inequality $s(\varphi) \le s$ cannot be replaced by equality.

4.4. **Twistor spaces of $G_2/\mathrm{SO}(4)$ as flags.** We saw in §2.1 that G_2 has three flag manifolds which fibre over $G_2/\mathrm{SO}(4)$; these give three twistor spaces for $G_2/\mathrm{SO}(4)$. To study these twistor spaces geometrically, we embed $G_2/\mathrm{SO}(4)$ in the oriented Grassmannian $\widetilde{G}_3(\mathbb{R}^7)$. The twistor spaces of this Grassmannian are flag manifolds $F = F^{\mathbb{R}}_{d_s,\dots,d_0}$ as described in the last subsection. Since $d_0 + 2(d_1 + \cdots + d_s) = 7$ and $\sum_{i>0, i \text{ odd}} d_i = 2$ there are precisely three possibilities which we will denote by F_1, F_2, F_3:

$$\text{(i) } s = 1 \text{ and } F_1 = F^{\mathbb{R}}_{2,3}\,; \quad \text{(ii) } s = 2 \text{ and } F_2 = F^{\mathbb{R}}_{1,2,1}\,; \quad \text{(iii) } s = 3 \text{ and } F_3 = F^{\mathbb{R}}_{1,1,1,1}\,.$$

Definition 4.9. Let $\psi = (\psi_{-s}, \dots, \psi_s) \in F_s$ $(s = 1, 2 \text{ or } 3)$. We shall say that ψ is a *G_2-flag* if

$$\psi_i \times \psi_j \subset \psi_{i+j} \quad \forall i, j \in \mathbb{Z}. \tag{4.9}$$

Here, we set $\psi_i = 0$ for $|i| > s$. We denote the set of all G_2-flags in F_s by $\mathcal{T}_s$.

Recall from §2.2 that the annihilator $\ell^a = \{X \in \mathrm{Im}\,\mathbb{O} : X \times \ell = 0\}$ of a 1-dimensional isotropic subspace ℓ of $\mathrm{Im}\,\mathbb{O}$ is isotropic, of dimension three, and contains ℓ. The following should be compared with [18, §5.2].

Proposition 4.10. *Let $\psi = (\psi_{-s}, \dots, \psi_s) \in F_s$ for some $s = 1, 2, 3$. The following are equivalent*:
(i) *ψ is a G_2-flag (Definition 4.9), i.e., $\psi \in \mathcal{T}_s$;*
(ii) *the ψ_i are the legs of a filtration*:

$$\begin{cases} (s=1) & \mathbb{C}^7 \supset \overline{W}^{\perp} \supset W \supset 0\,, \\ (s=2) & \mathbb{C}^7 \supset \overline{\ell}^{\perp} \supset \overline{(\ell^a)}^{\perp} \supset \ell^a \supset \ell \supset 0\,, \\ (s=3) & \mathbb{C}^7 \supset \overline{\ell}^{\perp} \supset \overline{D}^{\perp} \supset \overline{(\ell^a)}^{\perp} \supset \ell^a \supset D \supset \ell \supset 0\,, \end{cases} \tag{4.10}$$

where W is a 2-dimensional complex-coassociative subspace of $\mathbb{C}^7$, ℓ is a 1-dimensional isotropic subspace of $\mathbb{C}^7$, and D is a 2-dimensional subspace of ℓ^a which contains ℓ;
(iii) *ψ is G_2-equivalent to the flag*:

$$\begin{cases} (s=1) & (\ell_{-2\alpha_1-\alpha_2} \oplus \ell_{-\alpha_1-\alpha_2}\,,\, \ell_{-\alpha_1} \oplus \ell_0 \oplus \ell_{\alpha_1}\,,\, \ell_{\alpha_1+\alpha_2} \oplus \ell_{2\alpha_1+\alpha_2})\,, \\ (s=2) & (\ell_{-\alpha_1}\,,\, \ell_{\alpha_1+\alpha_2} \oplus \ell_{-2\alpha_1-\alpha_2}\,,\, \ell_0\,,\, \ell_{-\alpha_1-\alpha_2} \oplus \ell_{2\alpha_1+\alpha_2}\,,\, \ell_{\alpha_1})\,, \\ (s=3) & (\ell_{-2\alpha_1-\alpha_2}\,,\, \ell_{-\alpha_1}\,,\, \ell_{-\alpha_1-\alpha_2}\,,\, \ell_0\,,\, \ell_{\alpha_1+\alpha_2}\,,\, \ell_{\alpha_1}\,,\, \ell_{2\alpha_1+\alpha_2})\,. \end{cases} \tag{4.11}$$

Proof. We show that (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i).

First, suppose that (i) holds. When $s = 1$, put $W = \psi_1$; when $s = 2$, put $\ell = \psi_2$; when $s = 3$, put $\ell = \psi_3$ and $D = \psi_2 \oplus \psi_3$, then we get filtrations as in (4.10) so that (ii) holds.

Next, suppose that (ii) holds. Then by Lemma 2.3, we can find an element of G_2 which, when $s = 1$, maps W to the complex-coassociative subspace $\ell_{\alpha_1+\alpha_2} \oplus \ell_{2\alpha_1+\alpha_2}$; when $s = 2$, maps ℓ to the isotropic line ℓ_{α_1}, and when $s = 3$, maps ℓ and D to $\ell_{2\alpha_1+\alpha_2}$ and $\ell_{\alpha_1} \oplus \ell_{2\alpha_1+\alpha_2}$, respectively. The rest of the legs in (iii) then follow by calculation. For example, when $s = 3$, $(\ell_{2\alpha_1+\alpha_2})^a$ is spanned by ℓ_{α_1}, $\ell_{\alpha_1+\alpha_2}$ and $\ell_{2\alpha_1+\alpha_2}$, determining the legs ψ_i for i positive; the other legs follow by the reality condition (4.7). Hence (iii) holds.

Lastly, suppose that (iii) holds. Then simple calculations using (2.4) show that the displayed flags satisfy (4.9), so that (i) holds. □

Now, the projections $\pi_0 : F_s \to \widetilde{G}_3(\mathbb{R}^7)$ which combine even-numbered legs restrict to projections $\pi_0 : \mathcal{T}_s \to G_2/\mathrm{SO}(4)$. We now show that the resulting G_2-fibre bundles $\pi_0 : \mathcal{T}_s \to G_2/\mathrm{SO}(4)$ are isomorphic to the three twistor bundles $T_s \to G_2/\mathrm{SO}(4)$ defined in §2.1.

Proposition 4.11. *We have the following G_2-equivariant isomorphisms of fibre bundles over $G_2/\mathrm{SO}(4)$:*
$\mathcal{T}_1 \cong G_2/\mathrm{U}(2)_+$ *via* $\psi \mapsto W$ *where* $W = \psi_1$;
$\mathcal{T}_2 \cong Q^5$ *via* $\psi \mapsto \ell$ *where* $\ell = \psi_2$;
$\mathcal{T}_3 \cong G_2/(\mathrm{U}(1) \times \mathrm{U}(1))$ *via* $\psi \mapsto$ *the pair* (ℓ, D) *where* $\ell = \psi_2$ *and* $D = \mathrm{span}\{\psi_2, \psi_3\}$.

Proof. It suffices to describe the inverse mappings:
$G_2/\mathrm{U}(2)_+ \to \mathcal{T}_1$ given by $W \mapsto \psi =$ the legs of (4.10)(s=1);
$Q^5 \to \mathcal{T}_2$ given by $\ell \mapsto \psi =$ the legs of (4.10)(s=2);
$G_2/(\mathrm{U}(1) \times \mathrm{U}(1)) \to \mathcal{T}_3$ given by $(\ell, D) \mapsto \psi =$ the legs of (4.10)(s=3). □

Note that the legs of (4.10)(s=1) are $\psi = \big(\overline{W}, (W \oplus \overline{W})^\perp, W\big)$. This is the general form of the twistor lift of a strongly conformal map into a real Grassmannian [26, Example 6.16(ii)], with the additional restriction that we choose W positive, equivalently, complex-coassociative.

Note also that we have a G_2-equivariant isomorphism of fibre bundles $\mathcal{T}_2 \cong G_2/\mathrm{U}(2)_-$ via $\psi \mapsto W$ where $W = \psi_1$; with inverse $W \mapsto \psi =$ the legs of (4.10)(s=2) with $\ell = W \times W$. This commutes with the isomorphism given in §2.3.

Proposition 4.12. *The isomorphisms of Proposition 4.11 preserve horizontal spaces, J_1 and J_2.*

Proof. We shall prove that J_2 is preserved, as this is most important to us. Preservation of J_1 is similar. It follows that horizontal spaces are preserved.

Suppose that $\psi : M \to \mathcal{T}_s$ is a J_2-holomorphic map. Then various second fundamental forms A'_{ψ_i,ψ_j} vanish as in Proposition 4.3(ii). The J_2-holomorphicity conditions of Lemma 2.5 can be phrased as the vanishing of *some* of those. Thus, as a map into T_s, ψ is J_2-holomorphic.

The converses are trickier. We know some second fundamental forms vanish and we must show that all the others required by Proposition 4.3 vanish. We give the details for the three twistor spaces.

($s = 1$) Rephrasing Lemma 2.5 slightly, a map $W : M \to Z_1 = G_2/\mathrm{U}(2)_+$ is J_2-holomorphic if and only if (i) W is a holomorphic subbundle of $\varphi^\perp$, (ii) $W \subset \ker A'_{\varphi^\perp}$. Since $\psi_1 = W$, condition (i) means that the second fundamental form A'_{ψ_{-1},ψ_1} vanishes; condition (ii) means that the second fundamental form A'_{ψ_1,ψ_0} vanishes; by Lemma 4.1(i) and (ii), so does A'_{ψ_{-1},ψ_0}. By Proposition 4.3(ii), the vanishing of these three second fundamental forms is precisely the condition that the lift ψ given by (4.10)(s=1) be J_2-holomorphic.

($s = 2$) Again by Lemma 2.5, J_2-holomorphicity of ℓ is equivalent to (i) ℓ is a holomorphic subbundle of φ, (ii) $\ell \subset \ker A'_\varphi$. Since $\psi_2 = \ell$, condition (i) means that A'_{ψ_0,ψ_2} vanishes, by Lemma 4.1, so does A'_{ψ_{-2},ψ_0}; further by Lemma 4.1(iii), A'_{ψ_{-2},ψ_2} vanishes. Since ℓ is a holomorphic subbundle, $\psi_1 = \ell^a \ominus \ell$ is a holomorphic subbundle of $\varphi^\perp$ so that A'_{ψ_{-1},ψ_1} vanishes.

Next, condition (ii) tells us that A'_{ψ_2,ψ_1} and $A'_{\psi_2,\psi_{-1}}$ vanish, thus so do $A'_{\psi_{-1},\psi_{-2}}$ and $A'_{\psi_1,\psi_{-2}}$.

It remains to show that A'_{ψ_1,ψ_0} is zero. So suppose that it is non-zero. Then, since $\operatorname{rank}\psi_1 = 2$ and $\operatorname{rank}\psi_0 = 1$, $\ker A'_{\psi_1,\psi_0}$ has rank one. Thus we can choose a local basis a, b for ψ_1 with $b \in \ker A'_{\psi_1,\psi_0}$. By the G_2-condition (4.9), we have $a \times b \in \psi_2$. Applying A^φ_z gives $A^\varphi_z(a \times b) = A^\varphi_z(a) \times b + a \times A^\varphi_z(b) = A'_{\psi_1,\psi_0}(a) \times b$. Since $\psi_2 \subset \ker A^\varphi_z$, the left-hand side is zero. The right-hand side is the product of non-zero elements of ψ_0 and ψ_1; but calculating this from (4.11)(s=2), we see that such a product is non-zero, giving a contradiction.

Thus A'_{ψ_1,ψ_0} is zero, as is $A'_{\psi_0,\psi_{-1}}$, and ψ is J_2-holomorphic by Proposition 4.3.

($s = 3$) By Lemma 2.5, J_2-holomorphicity of (ℓ, D) is equivalent to (i) ℓ and D are holomorphic subbundles, (ii) $\ell \subset \ker A^\varphi_z$ and $A^\varphi_z(D) \subset \ell$.

Condition (i) implies holomorphicity of all the elements of the filtration (4.10)(s=3) which shows that A'_{ψ_i,ψ_j} is zero for $i - j$ even and $j > i$. Condition (ii) implies that A'_{ψ_i,ψ_j} is zero for $i = 3$ and so for $j = -3$, also for $i = 2$, $j = -3, -1, 1$ and so for $j = -2$, $i = -1, 1, 3$.

It remains to show that A'_{ψ_1,ψ_0} vanishes. To see this, from (4.9) we have $\psi_1 \times \psi_2 \subset \psi_3$. Applying A^φ_z gives $A^\varphi_z(\psi_1) \times \psi_2 = 0$ so that $\operatorname{Im} A'_{\psi_1,\psi_0} \subset \psi_2^a \cap \psi_0$. But, calculating this from (4.11)(s=3) we see that this intersection is trivial. Hence A'_{ψ_1,ψ_0}, and so also $A'_{\psi_0,\psi_{-1}}$, vanishes; ψ is J_2-holomorphic by Proposition 4.3. □

4.5. **Superhorizontal maps.** The concept of superhorizontal maps into a flag manifold is defined in [6]. For maps into a geometric flag manifold it amounts to

Definition 4.13. [6] A holomorphic map $\psi = (\psi_i)$ into a complex flag manifold is called *superhorizontal* if the only possible non-zero second fundamental forms A'_{ψ_i,ψ_j} occur when $j = i+1$.

Equivalently, for each $i = 0, 1, \ldots$, $\delta_i = \sum_{j=0}^{i} \psi_j$ is a holomorphic subbundle of $(\underline{\mathbb{C}}^n, \partial_{\bar{z}})$ and ∂_z maps sections of δ_i into sections of δ_{i+1}. By Proposition 4.3 we see that a superhorizontal map is both J_1-holomorphic and J_2-holomorphic with respect to π_e, and so horizontal, see [26] for more details.

The projection by π_e (or π_{odd}) of a superhorizontal map is a harmonic map of finite uniton number. In fact, let $\varphi : M \to G_2/\mathrm{SO}(4)$ have superhorizontal lift into T_s $(s \in \{1,2,3\})$. Then the extended solution [27] Φ of φ can be read off from the filtrations (4.10) namely, in terms of Segal's Grassmannian model [24]:

- $s(\varphi) = 1$, $\Phi\mathcal{H}_+ = \lambda^{-1}\overline{W} + W^{\perp} + \lambda\mathcal{H}_+$,
- $s(\varphi) = 2$, $\Phi\mathcal{H}_+ = \lambda^{-2}\overline{\ell} + \lambda^{-1}\overline{\ell^a} + (\ell^a)^{\perp} + \lambda\ell^{\perp} + \lambda^2\mathcal{H}_+$,
- $s(\varphi) = 3$, $\Phi\mathcal{H}_+ = \lambda^{-3}\overline{\ell} + \lambda^{-2}\overline{D} + \lambda^{-1}\overline{\ell}^a + +(\ell^a)^{\perp} + \lambda D^{\perp} + \lambda^2\ell^{\perp} + \lambda^3\mathcal{H}_+$.

These agree with the formulae of Correia and Pacheco [10], modulo conventions.

5. Relationship with almost complex maps to the 6-sphere

5.1. **The almost complex structure on S^6.** Let S^6 be the unit sphere in $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$. Define an almost complex structure J on S^6 by $J_F v = F \times v$ $(F \in S^6, v \in T_F S^6 = \mathrm{span}\{F\}^{\perp})$.

Lemma 5.1. *Let $F \in S^6$ and let $\alpha, \beta \in T_F^{1,0}S^6$. Then*

(i) $\alpha \times \overline{\beta} = \mathrm{i}\,(\alpha, \overline{\beta})F$;
(ii) $\alpha \times \beta \in T_F^{0,1}S^6$;
(iii) $|\alpha \times \beta|^2 = 2\big(|\alpha|^2|\beta|^2 - |(\alpha, \overline{\beta})|^2\big)$.

Proof. (i) From (2.2) we have

$$F \times (\alpha \times \overline{\beta}) = -\mathrm{i}\alpha \times \overline{\beta} - (\alpha, \overline{\beta})F. \tag{5.1}$$

However, we also have

$$F \times (\alpha \times \overline{\beta}) = -F \times (\overline{\beta} \times \alpha = -\{\mathrm{i}\overline{\beta} \times \alpha - (\alpha, \overline{\beta})F\} = \mathrm{i}\alpha \times \overline{\beta} + (\alpha, \overline{\beta})F.$$

We thus conclude that $F \times (\alpha \times \overline{\beta}) = 0$; since F is real, this means $\alpha \times \overline{\beta}$ is a multiple of F; by (2.1), we get $\alpha \times \overline{\beta} = \mathrm{i}(\alpha, \overline{\beta})F$.

(ii) From (2.2) we have

$$F \times (\alpha \times \beta) = -(F \times \alpha) \times \beta = -\mathrm{i}(\alpha \times \beta),$$

which establishes the result.

(iii) From (2.2) and since $T_F^{1,0}S^6$ is isotropic, we have

$$\begin{aligned}
(\alpha \times \beta\,,\, \overline{\alpha} \times \overline{\beta}) &= -\big(\beta\,,\, \alpha \times (\overline{\alpha} \times \overline{\beta})\big) \\
&= -\big(\beta\,,\, -(\alpha \times \overline{\alpha}) \times \overline{\beta} + 2(\alpha, \overline{\beta})\overline{\alpha} - |\alpha|^2\overline{\beta} - (\overline{\alpha}, \overline{\beta})\alpha\big) \\
&= -\big(\beta\,,\, -2|\alpha|^2\overline{\beta} + 2(\alpha, \overline{\beta})\overline{\alpha}\big) \\
&= 2\big\{|\alpha|^2|\beta|^2 - |(\alpha, \overline{\beta})|^2\big\}.
\end{aligned}$$

□

We can improve Lemma 5.1(ii) by using Lie theory, as follows. By transitivity of G_2 on S^6 we can take F in the zero weight space ℓ_0. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_2$. Consider now the copy of $\mathfrak{su}(3) \subset \mathfrak{g}_2$ given by the long roots:

$$\mathfrak{su}(3)^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + \mathfrak{g}_{\pm\alpha_2} + \mathfrak{g}_{\pm(3\alpha_1+\alpha_2)} + \mathfrak{g}_{\pm(3\alpha_1+2\alpha_2)}.$$

Under the action of $\mathfrak{su}(3)$, $\mathbb{C}^7$ decomposes as

$$\mathbb{C}^7 = (\ell_{-\alpha_1} + \ell_{-\alpha_1-\alpha_2} + \ell_{2\alpha_1+\alpha_2}) + \ell_0 + (\ell_{\alpha_1} + \ell_{\alpha_1+\alpha_2} + \ell_{-2\alpha_1-\alpha_2}). \tag{5.2}$$

Since $\mathfrak{su}(3)$ acts trivially on ℓ_0, it will also preserve the $(1,0)$- and $(0,1)$-spaces of the induced orthogonal complex structure on $\ell_0^{\perp}$, which therefore correspond to the two isotropic spaces in brackets in (5.2); those spaces correspond to the standard representation of $\mathfrak{su}(3)$ and its dual. Hence, we can take

$$T_F^{1,0}S^6 = \ell_{\alpha_1} + \ell_{\alpha_1+\alpha_2} + \ell_{-2\alpha_1-\alpha_2},$$

which shows us that $T_F^{1,0}S^6 \times T_F^{1,0}S^6$ *is equal to* $T_F^{0,1}S^6$.

5.2. **Almost complex maps into S^6.** Given a smooth map $F : M \to S^6$ from a surface into the 6-sphere, let $f : M \to \mathbb{R}P^6$ be its composition with the standard double covering $S^6 \to \mathbb{R}P^6$; equivalently, $f = \operatorname{span}\{F\}$ is the real line subbundle of $\underline{\mathbb{C}}^7$ of which F is a section. Embed $\mathbb{R}P^6$ in $\mathbb{C}P^6$. Then if F, equivalently f, is harmonic, we can define the Gauss bundles $G^{(i)}(f)$ $(i \in \mathbb{Z})$ as subbundles of the trivial bundle $\underline{\mathbb{C}}^7 = M \times \mathbb{C}^7$ or, equivalently, maps $M \to \mathbb{C}P^6$; these are harmonic and $G^{(-i)}(f) = \overline{G^{(i)}(f)}$ $(i \in \mathbb{Z})$ cf. §4.1.

The *isotropy order* of f (or F) is the maximum r such that $f \perp G^{(i)}(f)$ $(i = 1, 2, \ldots, r)$, or equivalently (cf. [8, Lemma 3.1]), such that any $r+1$ consecutive Gauss bundles $G^{(i)}(f)$ are zero or orthogonal (this condition is called $(r+1)$-*orthogonality* in, for example, [2]). The isotropy order of a harmonic map into S^n or $\mathbb{R}P^n$ is always odd, indeed, if it is at least $2s$, then $\big(G^{(s)}(f), G^{(s)}(f)\big) = 0$. Applying A_z^{φ} shows that the Gauss bundles $G^{(i)}(f)$ $i = -s, \ldots, s+1$ are orthogonal so that the isotropy order is at least $2s+1$. The map f is called *superminimal* (or (*real*) *isotropic*, or *pseudoholomorphic* [13] if it has infinite isotropy order, i.e., $G^{(i)}(f) \perp G^{(j)}(f)$ for all $i, j \in \mathbb{Z}$. This happens as soon as the isotropy order is at least n.

A map $M \to S^6$ is called *almost complex* if it is (almost-)holomorphic with respect to J, i.e., its differential intertwines the complex structure on M with J. Such maps are weakly conformal and harmonic, see, for example, [1]. Note that, for maps into $\mathbb{C}P^{n-1} = G_1(\mathbb{C}^n)$, the notions of strong and weak conformality coincide; in particular almost complex maps $M \to S^6$ are nilconformal.

The next result follows from work of [1] and [2]. We sketch a proof as we shall need some details from it. We use diagrams in the sense of [8].

Proposition 5.2. *Let $F : M \to S^6$ be a non-constant almost complex map. Write $f = \operatorname{span}\{F\}$. Then* either

(i) *F is a weakly conformal map into a totally geodesic $S^2 = \Pi^3 \cap S^6$ where Π^3 is an associative 3-dimensional subspace of $\mathbb{R}^7$*, or
(ii) *there is a G_2-flag*

$$\psi = (\psi_{-3}, \psi_{-2}, \psi_{-1}, \psi_0, \psi_1, \psi_2, \psi_3) : M \to G_2/(\mathrm{U}(1) \times \mathrm{U}(1))$$

with $\psi_i = G^{(i)}(f)$ $(i = -2, \ldots, 2)$ and we have the following diagram showing the only possible non-zero second fundamental forms A'_{ψ_i, ψ_j} :

$$\begin{tikzcd}
\psi_{-3} \arrow[r] & \psi_{-2} \arrow[r] & \psi_{-1} \arrow[r] & \psi_0 \arrow[r] & \psi_1 \arrow[r] & \psi_2 \arrow[r] \arrow[lllll, bend left] & \psi_3 \arrow[lllll, bend right]
\end{tikzcd} \tag{5.3}$$

Proof. Define F_i iteratively by $F_0 = F$, $F_i = A'_{G^{(i-1)}(f)}(F_{i-1})$, and $F_{-i} = \overline{F}_i$ $(i = 1, 2, \ldots)$, so that the F_i are sections of $G^{(i)}(f)$. Since F is non-constant, the sections $F_{\pm 1}$ are not identically zero, however, F_i may be zero when $|i|$ is sufficiently large. Since F is weakly conformal, it has isotropy order at least 3 (recall the isotropy order is always odd), so that $\{F_{-1}, F, F_1\}$ are mutually orthogonal. Since F is almost complex, we have

$$F \times F_1 = \mathrm{i} F_1\,. \tag{5.4}$$

(i) Suppose that F_2 is identically zero. Then $\Pi^3 = \mathrm{span}\{F_{-1}, F, F_1\}$ is a constant 3-dimensional subspace. By (5.4) and Lemma 2.1, it is associative, so that $F : M \to S^6$ is a weakly conformal map into the totally geodesic S^2 given by $S^2 = \Pi^3 \cap S^6$.

(ii) Suppose instead that F_2 is not identically zero. Then differentiating (5.4) gives

$$F \times F_2 = \mathrm{i} F_2 \,. \tag{5.5}$$

From (5.4) and (5.5), F_1 and F_2 both lie in the isotropic subspace $T^{1,0}_F S^6$; it follows that F has isotropy order at least 5.

Set $\Psi_i = F_i \ \ (i = -2, \dots, 2)$ and set

$$\Psi_3 = \Psi_1 \times \Psi_2 \,, \qquad \Psi_{-3} = \overline{\Psi_3}. \tag{5.6}$$

By Lemma 5.1, Ψ_3 is a non-zero element of $T^{0,1}_F S^6$ so that Ψ_{-3} is a non-zero element of $T^{1,0}_F S^6$. From (5.6), Ψ_{-3} is orthogonal to Ψ_1 and Ψ_2. Set $\psi_i = \mathrm{span}\{\Psi_i\} \ \ (i = -3, \dots, 3)$ and $\psi_i = 0$ for $|i| > 3$. Then the bundles ψ_i are mutually orthogonal, $T^{1,0}_F S^6 = \psi_1 \oplus \psi_2 \oplus \psi_{-3}$ and $T^{0,1}_F S^6 = \psi_{-1} \oplus \psi_{-2} \oplus \psi_3$.

We show that (ψ_i) is a G_2-flag, i.e., $\psi_i \times \psi_j = \psi_{i+j} \ \ (i, j \in \mathbb{Z})$. Indeed, $\Psi_1 \times \Psi_2 = \Psi_3$ implies that $\Psi_1 \times \Psi_3 = \Psi_1 \times (\Psi_1 \times \Psi_2) = 0$. Similarly $\Psi_2 \times \Psi_3 = 0$. Hence, setting $\ell = \psi_3$, and $D = \psi_3 \oplus \psi_2$, the ψ_i are the legs of the standard G_2-flag (4.11)($s = 3$).

Remark 5.3. In [2, page 147] there is a precise multiplication table for the Ψ_i.

We show that we get the diagram (5.3). First $\psi_{-2} \to \cdots \to \psi_2$ is part of the harmonic sequence of f so there are no other second fundamental forms between these elements. Next $\mathrm{Im}\, A'_{\psi_2} = G'(G^{(2)}(f)) = G^{(3)}(f)$. Since f has isotropy order at least 5, this is orthogonal to $\psi_{-2} \oplus \cdots \oplus \psi_2$, hence $\mathrm{Im}\, A'_{\psi_2} \in \psi_{-3} \oplus \psi_3$; taking the conjugate, $\mathrm{Im}\, A''_{\psi_{-2}} \in \psi_{-3} \oplus \psi_3$. By Lemma 4.1(iii), $A'_{\psi_3, -\psi_3} = 0$. It follows that the only possible non-zero second fundamental forms are those shown in (5.3). □

Remark 5.4. In [1], almost complex maps $F : M \to S^6$ are classified into four types as follows: (I) full superminimal harmonic maps into S^6, (II) F full non-superminimal harmonic maps into S^6, (III) F full non-superminimal harmonic maps into a totally geodesic S^5, (IV) weakly conformal maps into $S^2 = \Pi^3 \cap S^6$ for some associative 3-dimensional subspace Π^3.

For a Type I map, diagram (5.3) reduces to the harmonic sequence:

$$G^{(-3)}(f) \to G^{(-2)}(f) \to G^{(-1)}(f) \to f \to G^{(1)}(f) \to G^{(2)}(f) \to G^{(3)}(f), \tag{5.7}$$

see also §5.4. For a Type II map, all the second fundamental forms in diagram (5.3) are non-zero. For a Type III map, the harmonic sequence of f lies in a 6-dimensional subspace of $\mathbb{R}^7$ and is periodic of period 6.

5.3. **Building harmonic maps into $G_2/\mathrm{SO}(4)$.** We give a way of building harmonic maps into $G_2/\mathrm{SO}(4)$ from almost complex maps into S^6. We need the following *Reduction Theorem.*

Theorem 5.5. (a) [8, Theorem 4.1] *Let f be a harmonic map from a surface to a complex Grassmannian, and let α be* (i) *a holomorphic line subbundle of* $\ker A'_{f^\perp}$, *or* (ii) *an antiholomorphic line subbundle of* $\ker A''_{f^\perp}$, *then $f \oplus \alpha$ is also harmonic.*

(b) *$f \oplus \alpha$ is nilconformal if and only if f is.*

Proof. (a) This is another example (cf. §4.1) of adding a uniton [27], or see [8].

(b) It is easily checked that both $(A^\varphi_z)^{2k+1}(\varphi^\perp) = (A^f_z)^{2k+1}(f^\perp)$ for any $k \in \mathbb{N}$. The result follows. □

Proposition 5.6. *Let $F : M \to S^6$ be almost complex and let α be a holomorphic line subbundle of $F^{-1}T^{1,0}S^6$. Set*

$$\varphi = \overline{\alpha} \oplus \mathrm{span}\{F\} \oplus \alpha. \tag{5.8}$$

Then φ is a nilconformal harmonic map from M to $G_2/\mathrm{SO}(4)$.

Proof. Let L be a nowhere zero (local) section of α. By Lemma 5.1, $L \times \overline{L}/|L|^2 = iF$, then by Lemma 2.1, φ is associative. From the proof of Proposition 5.2, $F^{-1}T^{1,0}S^6 = \psi_{-3} \oplus \psi_1 \oplus \psi_2 \subset (\psi_{-1} \oplus \psi_0)^{\perp} \subset \ker A'_{f^{\perp}}$. Thus $\alpha \subset \ker A'_{f^{\perp}}$ and by Theorem 5.5 part (i), $\widehat{\varphi} = f \oplus \alpha$ is a harmonic map into a Grassmannian.

Now $\overline{\alpha}$ lies in $\ker A''_{f^{\perp}}$; further, by Lemma 4.1(iii), $A''_{\overline{\alpha},\alpha} = 0$. Hence $\overline{\alpha}$ lies in $\ker A''_{\widehat{\varphi}^{\perp}}$. Hence, we may apply Theorem 5.5 part (ii) to see that $\varphi = \widehat{\varphi} \oplus \overline{\alpha}$ is harmonic. $\square$

Remark 5.7. Since φ is obtained from f by adding a couple of unitons, the harmonic map (5.8) is of finite uniton number if and only the almost complex map $F : M \to S^6$ is of finite uniton number.

Example 5.8. Suppose that F is constant. Then $F^{-1}T^{1,0}S^6 = M \times T^{1,0}_F S^6$ is a constant maximally isotropic subbundle of $f^{\perp} \otimes \mathbb{C} \cong \mathbb{C}^6$. Choose an identification of the vector space $T^{1,0}_F S^6$ with $\mathbb{C}^3$. Then, given a non-constant holomorphic map $\alpha : M \to \mathbb{C}P^2$, we get a corresponding line subbundle α of $F^{-1}T^{1,0}S^6$, and Proposition 5.6 gives a harmonic map $\varphi : M \to G_2/\mathrm{SO}(4)$.

In fact, it can be checked that (a) φ and $\varphi^{\perp}$ are strongly conformal, (b) $\operatorname{rank} G'(\varphi) = 1$, and (c) $G'(\varphi^{\perp}) \times G''(\varphi^{\perp})$ is a constant subbundle (namely f). The construction gives a one-to-one correspondence between pairs (F, α) as above and harmonic maps φ satisfying these three properties.

In the case that F is non-constant, by setting $\alpha = G'(f)$ we obtain harmonic maps into $G_2/\mathrm{SO}(4)$ which complement those of Example 5.8 as follows. Recall that a map φ into a Grassmannian is strongly conformal if and only if $s(\varphi) = 1$.

Theorem 5.9. *There is a one-to-one correspondence between*

(1) *almost complex maps $F : M \to S^6$ with image not contained in a totally geodesic S^2, and*
(2) *strongly conformal harmonic maps $\varphi : M \to G_2/\mathrm{SO}(4)$ with* (a) *$\varphi^{\perp}$ strongly conformal,* (b) $\operatorname{rank} G'(\varphi) = 1$ *and* (c) *$G'(\varphi^{\perp}) \times G''(\varphi^{\perp})$ a non-constant subbundle,*

given by

$$F \mapsto \varphi = G''(f) \oplus f \oplus G'(f) \quad \textit{with inverse} \quad \varphi \mapsto F = \mathrm{i}\overline{L} \times L/|L|^2, \tag{5.9}$$

where $f = \operatorname{span}\{F\}$ and L is any nowhere zero (local) section of $G'(\varphi^{\perp})$.

Proof. (i) Let F be as in (1). Since $f = \operatorname{span}\{F\}$ is non-constant, $G^{(-1)}(f)$, f and $G^{(1)}(f)$ are non-zero and mutually orthogonal. Since f is almost complex, by Lemma 2.1, $\varphi = G^{(-1)}(f) \oplus f \oplus G^{(1)}(f)$ is a map into $G_2/\mathrm{SO}(4)$. We have case (ii) of Proposition 5.2, and we use the notation of that proposition writing $\psi_i = G^{(i)}(f)$ $\;(|i| \leq 2)$. we have $G'(\varphi) = G^{(2)}(f) = \psi_2$ which is non-zero and isotropic, so that φ is non-constant and strongly conformal with $\operatorname{rank} G'(\varphi) = 1$. In fact, combining subbundles in (5.3), we obtain the diagram

$$\begin{tikzcd} \psi_{-3} \oplus \psi_{-2} \arrow[r] & \psi_{-1} \oplus \psi_0 \oplus \psi_1 \arrow[r] & \psi_2 \oplus \psi_3 \arrow[ll, bend right] \end{tikzcd} \tag{5.10}$$

which shows that φ has the twistor lift $(\overline{W}, \varphi, W) : M \to T_1$ where $W = \psi_2 \oplus \psi_3$. Further, $G'(\varphi^{\perp}) = \psi_{-1}$ is isotropic, so that $\varphi^{\perp}$ is strongly conformal; also $G'(\varphi^{\perp}) \times G''(\varphi^{\perp}) = \psi_{-1} \times \psi_1 = f$, which is non-constant so condition (2)(c) holds

(ii) Conversely, let φ be as in (2) and set $\alpha = G''(\varphi^{\perp})$. Since $\varphi^{\perp}$ is strongly conformal, this is isotropic; also, by Lemma 4.1(i), $\operatorname{rank} \alpha = \operatorname{rank} A''_{\varphi^{\perp}} = \operatorname{rank} A'_{\varphi} = 1$. For any nowhere zero (local) section L of α, set $F = \mathrm{i}\overline{L} \times L/|L|^2$ — note that this is well defined under different choices of L — and set $f = \operatorname{span}\{F\}$. By condition (2)(c), f is non-constant. Further, we have an orthogonal decomposition: $\varphi = \overline{\alpha} \oplus f \oplus \alpha$.

Now f is orthogonal to $\operatorname{Im} A''_{\varphi^{\perp}}$, so is contained in $\ker A'_{\varphi}$; hence, $G'(f) \subset \varphi$. But $\overline{\alpha} = \operatorname{Im} A'_{\varphi^{\perp}}$ is a holomorphic subbundle of φ, so that $A''_{\overline{\alpha},f} = 0$; it follows that $G'(f) \subset \alpha$. Since f is non-constant, $G'(f) = \alpha$, so we have the decomposition $\varphi = G''(f) \oplus f \oplus G'(f)$. Further, by Lemma 2.1, $F \times L = \mathrm{i}L$ showing that F is almost complex.

It is easily checked that the two transformations in (5.9) are inverse. □

The above constructions give strongly conformal harmonic maps into $G_2/\mathrm{SO}(4)$ with Gauss bundles of rank 1. To get ones with Gauss bundle of rank 2, we adopt a different sort of construction, as follows.

Proposition 5.10. *Let $F : M \to S^6$ be an almost complex map with image not contained in a totally geodesic S^2. Set $\psi_3 = G^{(1)}(f) \times G^{(2)}(f)$ and $\psi_{-3} = \overline{\psi_3}$. Then $\varphi = \psi_{-3} \oplus f \oplus \psi_3$ is a strongly conformal harmonic map $M \to G_2/\mathrm{SO}(4)$ with* rank $G'(\varphi) = 2$.

Proof. Let (ψ_i) be the G_2-flag (5.3) generated by f, i.e., $\psi_i = G^{(i)}(f) \quad (i = -2, \dots, 2)$, $\psi_3 = \psi_1 \times \psi_2$, $\psi_{-3} = \overline{\psi_3}$. Then setting $\varphi = \psi_{-3} \oplus \psi_0 \oplus \psi_3$ and $W = \psi_1 \oplus \psi_{-2}$; by Lemma 5.1(i), φ is associative, W is complex-coassociative, and φ has J_2-holomorphic lift $(\overline{W}, \varphi, W)$. Further $G'(\varphi) = G'(\psi_0) + G'(\psi_{-3}) = W$ has rank 2. □

We remark that harmonicity of φ also follows from an extension of Proposition 5.6.

Example 5.11. Let $v_1 \in \ell_{\alpha_1}$, $v_2 \in \ell_{\alpha_1+\alpha_2}$ have norm $1/\sqrt{2}$ and set $v_3 = \overline{v_1 \times v_2}$. Then v_3 also has norm $1/\sqrt{2}$ and lies in $\ell_{-2\alpha_1-\alpha_2}$, and we see that $v_p \times v_q = \epsilon_{pqr} v_r$ and $v_p \times \overline{v}_q = (\mathrm{i}/2)\delta_{pq} L_0$ where $L_0 \in \ell_0$ has norm one. As in [1, p. 420], define a map $f : \mathbb{C} \to S^6$ by

$$f(z) = \frac{1}{\sqrt{3}} \sum_{j=1}^{3} \left(\mathrm{e}^{\mu_j z - \overline{\mu}_j \bar{z}} v_j + \mathrm{e}^{-\mu_j z + \overline{\mu}_j \bar{z}} \overline{v}_j \right) \tag{5.11}$$

where the μ_j are the three cube roots of unity. Then f is doubly periodic of periods π and $\mathrm{i}\pi/\sqrt{3}$ so factors to the torus $T^2 = \mathbb{C}/\langle \pi, \mathrm{i}\pi/\sqrt{3} \rangle$. Further, it is an almost complex map of type (III) into the S^5 orthogonal to L_0. By [2, Corollary 6.4], this map is of *finite type*, by [21, Theorem 8] it follows that it is not of finite uniton number.

Applying any of the constructions above to f gives nilconformal harmonic maps $\varphi : T^2 \to G_2/\mathrm{SO}(4)$. Since f is not of finite uniton number, by Remark 5.7, the maps φ are not of finite uniton number either.

5.4. **Superminimal harmonic maps.** Recall the diagram (2.5). We define the projection $\pi_6 : Q^5 \to S^6$ by $\pi_6(\ell) = iL \times \overline{L}/|L|^2$ for any nowhere zero section L of ℓ. The following definition is given by R. L. Bryant [4].

Definition 5.12. A full holomorphic map $h : M \to Q^5$ is called *superhorizontal* if it satisfies $h \times G'(h) = 0$, equivalently, any holomorphic section H of h satisfies $H \times H' = 0$.

Setting $\psi_i = G^{(3+i)}(h)$, we have a map $\widehat{\psi} : M \to F_2 = F^{\mathbb{R}}_{2,3}$ defined by $(\widehat{\psi}_{-2}, \widehat{\psi}_{-1}, \widehat{\psi}_0, \widehat{\psi}_1, \widehat{\psi}_2) = (\psi_{-3}, \psi_{-2} \oplus \psi_{-1}, \psi_0, \psi_1 \oplus \psi_2, \psi_3)$. Differentiating $h \times G'(h) = 0$ gives $h \times G^{(2)}(h) = 0$; it follows that *a full holomorphic map $h : M \to Q^5$ is superhorizontal if and only if $\widehat{\psi}$ coincides with the unique G_2-flag* (4.10)*(s=2) with $\ell = \overline{h}$*. Thus, under the identification of Q^5 with $\mathcal{T}_2$, a superhorizontal holomorphic map (Definition 5.12) $h : M \to Q^5$ corresponds to a superhorizontal holomorphic map (Definition 4.13) $\widehat{\psi} : M \to \mathcal{T}_2$. Note that the harmonic sequence $\psi = (\psi_{-3}, \psi_{-2}, \psi_{-1}, \psi_0, \psi_1, \psi_2, \psi_3)$ defines another superhorizontal map, this time into $\mathcal{T}_3$.

Let $F : M \to S^6$ be an almost complex map of type (I) (i.e., full and superminimal). Then its harmonic sequence (5.7) is a G_2-flag, in particular, $G^{(-3)}(f) \times G^{(-2)}(f) = 0$. It follows that the first leg $h = G^{(-3)}(f)$ of this flag is full, holomorphic, superhorizontal map into Q^5. We are led to the following result of Bryant [4] as described in [3].

Theorem 5.13. *There is a one-to-one correspondence between*

(1) *linearly full superhorizontal holomorphic maps $h : M \to Q^5$, and*
(2) *almost complex maps $F : M \to S^6$ of type (I),*

given by $F = \mathrm{i}(\overline{H} \times H)/|H|^2$, where H is any (local) section of h, with inverse $h = G^{(-3)}(f)$ where $f = \mathrm{span}\{F\}$.

Note that $f = G^{(3)}(h)$. For examples of full holomorphic superhorizontal maps $h : S^2 \to Q^5$ given by simple polynomial formulae, see [15, 20]. We can obtain harmonic maps into $G_2/\mathrm{SO}(4)$ from such h as follows.

Theorem 5.14. *Let $h : M \to Q^5$ be a linearly full superhorizontal holomorphic map and let F be the corresponding almost complex map. Then $\varphi = G^{(-i)}(f) \oplus f \oplus G^{(i)}(f)$ is a harmonic map into $G_2/\mathrm{SO}(4)$ for $i = 1, 2, 3$. Further, φ is strongly conformal (i.e. $s(\varphi) = 1$) for $i = 1, 3$ but $s(\varphi) = 3$ for $i = 2$.*

Proof. For $i = 1$, this is implied by Theorem 5.9.

For $i = 2$, $\varphi = \pi_3 \circ \psi$ where $\psi = (\psi_{-3}, \dots, \psi_3) : M \to \mathcal{T}_3$ is the harmonic sequence of h. Since this is certainly J_2-holomorphic, φ is harmonic.

For $i = 3$, this is as a special case of Proposition 5.10. □

Remark 5.15. When $i = 3$, φ has the J_2-holomorphic lift $(\psi_{-1} \oplus \psi_2, \varphi, \psi_1 \oplus \psi_{-2})$ into $\mathcal{T}_1$ as in Proposition 5.10; it also has the superhorizontal lift $(\psi_{-3}, \psi_{-2} \oplus \psi_{-1}, \psi_0, \psi_1 \oplus \psi_2, \psi_3)$ into $\mathcal{T}_2$.

The above construction doesn't give harmonic maps with $s(\varphi) = 2$. For these we need to use Theorem 5.6 with a careful choice of α as follows.

Example 5.16. Let $h : M \to Q^5$ be a linearly full superhorizontal holomorphic map and let F be the corresponding almost complex map. Set $f = \mathrm{span}\{F\}$ and let α be a holomorphic subbundle of $f^{\perp}$ which lies in $h \oplus G^{(4)}(h) = G^{(-3)}(f) \oplus G^{(1)}(f)$ but is not equal to either h or $G^{(4)}(h)$. Then the harmonic map defined by (5.8) has $s(\varphi) = 2$.

To see this, we may check that $(A^{\varphi}_{z})^4(\varphi^{\perp}) = 0$. Also $(A^{\varphi}_{z})^2(\varphi) \neq 0$, equivalently $G'(\varphi)$ is not isotropic: indeed, it contains $G'(\alpha)$ which has non-zero components in both $G^{(-2)}(f)$ and $G^{(2)}(f)$ and so cannot be isotropic.

In conclusion, we can find examples of harmonic maps from surfaces to $G_2/\mathrm{SO}(4)$ with twistor lifts into any of its three twistor spaces. If we start with the explicit polynomial formulae of [15, 20] alluded to above, we can obtain such examples with components defined by polynomials

References

[1] J. Bolton, L. Vrancken, and L. M. Woodward, *On almost complex curves in the nearly Kähler 6-sphere*, Quart. J. Math. Oxford Ser. (2) 45 (1994), no. 180, 407–427.

[2] J. Bolton, F. Pedit, and L. M. Woodward, *Minimal surfaces and the affine Toda field model*, J. Reine Angew. Math. **459** (1995), 119–150.

[3] J. Bolton and L. M. Woodward, *Special submanifolds of S^6 with its G_2 geometry*, Geometry, topology and physics (Campinas, 1996), 59–68, de Gruyter, Berlin, 1997.

[4] R.L. Bryant, *Submanifolds and special structures on the octonians*, J. Differential Geom. **17** (1982), 185–232.

[5] F. E. Burstall, *A twistor description of harmonic maps of a 2-sphere into a Grassmannian*, Math. Ann. **274** (1986), 61–74.

[6] F. E. Burstall and J. H. Rawnsley, *Twistor theory for Riemannian symmetric spaces*, Lecture Notes in Mathematics, 1424, Springer-Verlag, Berlin, Heidelberg, 1990.

[7] F. E. Burstall and S. M. Salamon, *Tournaments, flags, and harmonic maps*, Math. Ann. **277** (1987), 249–265.

[8] F. E. Burstall and J. C. Wood, *The construction of harmonic maps into complex Grassmannians*, J. Diff. Geom. **23** (1986), 255–298.

[9] J. Cheeger and D. G. Ebin, *Comparison theorems in Riemannian geometry*, Revised reprint of the 1975 original, AMS Chelsea Publishing, Providence, RI, 2008.

[10] N. Correia and R. Pacheco, *Harmonic maps of finite uniton number into G_2*, Math. Z. **271** (2012), 13–32.

[11] J. Davidov and A. G. Sergeev, *Twistor spaces and harmonic maps* (Russian), Uspekhi Mat. Nauk **48** (1993), no. 3 (291), 3–96; translation in Russian Math. Surveys **48** (1993), no. 3, 1–91.

[12] J. Eells and L. Lemaire, *Another report on harmonic maps*, Bull. London Math. Soc. **20** (1988), 385–524.

[13] J. Eells and J. C. Wood, *Harmonic maps from surfaces to complex projective spaces*, Advances in Math. **49** (1983), 217–263.

[14] S. Erdem and J. C. Wood, *On the constructions of harmonic maps into a Grassmannian*, J. London Math. Soc. (2), **28** (1983), 161–174.

[15] L. Fernández, *The space of almost complex 2-spheres in the 6-sphere*, Trans. Amer. Math. Soc. (to appear).

[16] W. Fulton and J. Harris, *Representation theory. A first course*, Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, 1991.

[17] R. Harvey and H. B. Lawson, *Calibrated geometries*, Acta Math. **148** (1982), 47–157.

[18] P.Z. Kobak, *Quaternionic Geometry and Harmonic Maps*, thesis, University of Oxford (1993).

[19] J. L. Koszul and B. Malgrange, *Sur certaines structures fibrées complexes*, Arch. Math. **9** (1958), 102–109.

[20] J. K. Martins, *Superminimal surfaces in the* 6*-sphere*, Bull. Braz. Math. Soc., (N.S) **44**, no. 1, 1–24.

[21] R. Pacheco, *On harmonic tori in compact rank one symmetric spaces*, Diff. Geom. Appl. **27** (2009), 352–361.

[22] S. Salamon, *Quaternionic Kähler manifolds*, Invent. Math. **67** (1982), 143–171.

[23] S. Salamon, *Riemannian geometry and holonomy groups*, Pitman Research Notes in Mathematics Series, 201, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989.

[24] G. Segal, *Loop groups and harmonic maps*, Advances in homotopy theory (Cortona, 1988), 153–164, London Math. Soc. Lecture Notes Ser., 139, Cambridge Univ. Press, Cambridge, 1989.

[25] M. Svensson and J. C. Wood, *Filtrations, factorizations and explicit formulae for harmonic maps*, Comm. Math. Phys. **310** (2012), 99–134.

[26] M. Svensson and J. C. Wood, *New constructions of twistor lifts for harmonic maps*, arXiv:1106.1832 (2012).

[27] K. Uhlenbeck, *Harmonic maps into Lie groups: classical solutions of the chiral model*, J. Differential Geom. **30** (1989), 1–50.

[28] J. G. Wolfson, *Harmonic sequences and harmonic maps of surfaces into complex Grassmann manifolds*, J. Differential Geom. **27** (1988), 161–178.

[29] J. C. Wood, *Explicit constructions of harmonic maps*, in: Harmonic Maps and Differential Geometry, ed. E. Loubeau and S. Montaldo, Contemp. Math., 542, Amer. Math. Soc. (2011), 41–74.

Department of Mathematics & Computer Science, University of Southern Denmark, and CP3–Origins, Centre of Excellence for Particle Physics and Phenomenology, Campusvej 55, DK-5230 Odense M, Denmark

E-mail address: svensson@imada.sdu.dk

Department of Pure Mathematics, University of Leeds, Leeds LS2 9JT, Great Britain

E-mail address: j.c.wood@leeds.ac.uk